MATH 311-504

Topics in Applied Mathematics

Lecture 3-5:
Orthogonal bases.
The Gram-Schmidt orthogonalization process.

Inner product

The notion of *inner product* generalizes the notion of dot product of vectors in $\mathbb{R}^n$.

Definition. Let V be a vector space. A function $\beta : V \times V \to \mathbb{R}$, usually denoted $\beta(\mathbf{x}, \mathbf{y}) = \langle \mathbf{x}, \mathbf{y} \rangle$, is called an **inner product** on V if it is *positive, symmetric*, and *bi-linear*. That is, if

(i) $\langle \mathbf{x}, \mathbf{y} \rangle \geq 0$, $\langle \mathbf{x}, \mathbf{x} \rangle = 0$ only for $\mathbf{x} = \mathbf{0}$ (positivity)
(ii) $\langle \mathbf{x}, \mathbf{y} \rangle = \langle \mathbf{y}, \mathbf{x} \rangle$ (symmetry)
(iii) $\langle r\mathbf{x}, \mathbf{y} \rangle = r\langle \mathbf{x}, \mathbf{y} \rangle$ (homogeneity)
(iv) $\langle \mathbf{x} + \mathbf{y}, \mathbf{z} \rangle = \langle \mathbf{x}, \mathbf{z} \rangle + \langle \mathbf{y}, \mathbf{z} \rangle$ (additivity)

Bi-linearity means that $\langle \mathbf{x}, \mathbf{y} \rangle$ is linear both as a function of $\mathbf{x}$ and as a function of $\mathbf{y}$.

Principal examples.

(a) Dot product: $V = \mathbb{R}^n$,

$$\langle \mathbf{x}, \mathbf{y} \rangle = \mathbf{x} \cdot \mathbf{y} = x_1y_1 + x_2y_2 + \cdots + x_ny_n.$$

(b) $V = C[a, b]$,

$$\langle f, g \rangle = \int_a^b f(x)g(x)\,dx.$$

Norm

Theorem Suppose $\langle \mathbf{x}, \mathbf{y} \rangle$ is an inner product on a vector space V. Then $\|\mathbf{x}\| = \sqrt{\langle \mathbf{x}, \mathbf{x} \rangle}$ is a **norm** on V, which means that the following conditions hold:

(i) $\|\mathbf{x}\| \geq 0$, $\|\mathbf{x}\| = 0$ only for $\mathbf{x} = \mathbf{0}$ (positivity)

(ii) $\|r\mathbf{x}\| = |r| \, \|\mathbf{x}\|$ for all $r \in \mathbb{R}$ (homogeneity)

(iii) $\|\mathbf{x} + \mathbf{y}\| \leq \|\mathbf{x}\| + \|\mathbf{y}\|$ (triangle inequality)

Examples. • If $\langle \mathbf{x}, \mathbf{y} \rangle = \mathbf{x} \cdot \mathbf{y}$ then $\|\mathbf{x}\| = |\mathbf{x}|$.

• If $\langle f, g \rangle = \int_a^b f(x)g(x)\, dx$, then

$$\|f\| = \left(\int_a^b |f(x)|^2 \, dx \right)^{1/2}.$$

Angle

Cauchy-Schwarz Inequality:

$$|\langle \mathbf{x}, \mathbf{y} \rangle| \leq \sqrt{\langle \mathbf{x}, \mathbf{x} \rangle} \sqrt{\langle \mathbf{y}, \mathbf{y} \rangle} = \|\mathbf{x}\| \, \|\mathbf{y}\|.$$

As a consequence, we can define the *angle* between vectors in any vector space with an inner product (and induced norm):

$$\angle(\mathbf{x}, \mathbf{y}) = \arccos \frac{\langle \mathbf{x}, \mathbf{y} \rangle}{\|\mathbf{x}\| \, \|\mathbf{y}\|}$$

so that

$$\langle \mathbf{x}, \mathbf{y} \rangle = \|\mathbf{x}\| \, \|\mathbf{y}\| \cos \angle(\mathbf{x}, \mathbf{y}).$$

In particular, vectors $\mathbf{x}$ and $\mathbf{y}$ are *orthogonal* (denoted $\mathbf{x} \perp \mathbf{y}$) if $\langle \mathbf{x}, \mathbf{y} \rangle = 0$.

Orthogonal systems

Let V be an inner product space with an inner product $\langle \cdot, \cdot \rangle$ and the induced norm $\| \cdot \|$.

Definition. A nonempty set $S \subset V$ is called an **orthogonal system** if all vectors in S are mutually orthogonal. That is, $\langle \mathbf{x}, \mathbf{y} \rangle = 0$ for any $\mathbf{x}, \mathbf{y} \in S$, $\mathbf{x} \neq \mathbf{y}$.

An orthogonal system $S \subset V$ is called **orthonormal** if $\|\mathbf{x}\| = 1$ for any $\mathbf{x} \in S$.

Theorem Any orthogonal system without zero vector is a linearly independent set.

Examples. • $V = \mathbb{R}^n$, $\langle \mathbf{x}, \mathbf{y} \rangle = \mathbf{x} \cdot \mathbf{y}$.

The standard basis $\mathbf{e}_1 = (1, 0, 0, \dots, 0)$, $\mathbf{e}_2 = (0, 1, 0, \dots, 0)$, ..., $\mathbf{e}_n = (0, 0, 0, \dots, 1)$ forms an orthonormal system.

• $V = \mathbb{R}^3$, $\langle \mathbf{x}, \mathbf{y} \rangle = \mathbf{x} \cdot \mathbf{y}$.

$\mathbf{v}_1 = (3, 5, 4)$, $\mathbf{v}_2 = (3, -5, 4)$, $\mathbf{v}_3 = (4, 0, -3)$.

This set is orthogonal but not orthonormal.

• $V = C[-\pi, \pi]$, $\langle f, g \rangle = \dfrac{1}{\pi} \displaystyle\int_{-\pi}^{\pi} f(x) g(x)\, dx$.

$f_1(x) = \sin x$, $f_2(x) = \sin 2x$, ..., $f_n(x) = \sin nx$, ...

This is an orthonormal system.

Orthonormal bases

Let $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$ be an orthonormal basis for an inner product space V.

Theorem Let $\mathbf{x} = x_1\mathbf{v}_1 + x_2\mathbf{v}_2 + \cdots + x_n\mathbf{v}_n$ and $\mathbf{y} = y_1\mathbf{v}_1 + y_2\mathbf{v}_2 + \cdots + y_n\mathbf{v}_n$, where $x_i, y_j \in \mathbb{R}$. Then

(i) $\langle \mathbf{x}, \mathbf{y} \rangle = x_1y_1 + x_2y_2 + \cdots + x_ny_n$,

(ii) $\|\mathbf{x}\| = \sqrt{x_1^2 + x_2^2 + \cdots + x_n^2}$.

Proof: (ii) follows from (i) when $\mathbf{y} = \mathbf{x}$.

$$\langle \mathbf{x}, \mathbf{y} \rangle = \left\langle \sum_{i=1}^{n} x_i\mathbf{v}_i, \sum_{j=1}^{n} y_j\mathbf{v}_j \right\rangle = \sum_{i=1}^{n} x_i \left\langle \mathbf{v}_i, \sum_{j=1}^{n} y_j\mathbf{v}_j \right\rangle$$

$$= \sum_{i=1}^{n}\sum_{j=1}^{n} x_iy_j\langle \mathbf{v}_i, \mathbf{v}_j \rangle = \sum_{i=1}^{n} x_iy_i.$$

Let $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$ be a basis for an inner product space V.

Theorem If the basis $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$ is an orthogonal set then for any $\mathbf{x} \in V$

$$\mathbf{x} = \frac{\langle \mathbf{x}, \mathbf{v}_1 \rangle}{\langle \mathbf{v}_1, \mathbf{v}_1 \rangle}\mathbf{v}_1 + \frac{\langle \mathbf{x}, \mathbf{v}_2 \rangle}{\langle \mathbf{v}_2, \mathbf{v}_2 \rangle}\mathbf{v}_2 + \cdots + \frac{\langle \mathbf{x}, \mathbf{v}_n \rangle}{\langle \mathbf{v}_n, \mathbf{v}_n \rangle}\mathbf{v}_n.$$

If $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$ is an orthonormal set then

$$\mathbf{x} = \langle \mathbf{x}, \mathbf{v}_1 \rangle \mathbf{v}_1 + \langle \mathbf{x}, \mathbf{v}_2 \rangle \mathbf{v}_2 + \cdots + \langle \mathbf{x}, \mathbf{v}_n \rangle \mathbf{v}_n.$$

Proof: We have that $\mathbf{x} = x_1\mathbf{v}_1 + \cdots + x_n\mathbf{v}_n$.

$\implies \langle \mathbf{x}, \mathbf{v}_i \rangle = \langle x_1\mathbf{v}_1 + \cdots + x_n\mathbf{v}_n, \mathbf{v}_i \rangle, \; 1 \le i \le n.$

$\implies \langle \mathbf{x}, \mathbf{v}_i \rangle = x_1\langle \mathbf{v}_1, \mathbf{v}_i \rangle + \cdots + x_n\langle \mathbf{v}_n, \mathbf{v}_i \rangle$

$\implies \langle \mathbf{x}, \mathbf{v}_i \rangle = x_i\langle \mathbf{v}_i, \mathbf{v}_i \rangle.$

Orthogonal projection

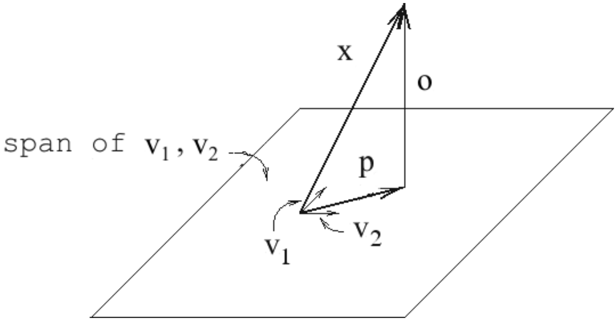
Let V be an inner product space.

Let $\mathbf{x}, \mathbf{v} \in V$, $\mathbf{v} \neq \mathbf{0}$. Then $\mathbf{p} = \dfrac{\langle \mathbf{x}, \mathbf{v} \rangle}{\langle \mathbf{v}, \mathbf{v} \rangle} \mathbf{v}$ is the **orthogonal projection** of the vector $\mathbf{x}$ onto the vector $\mathbf{v}$. That is, the remainder $\mathbf{o} = \mathbf{x} - \mathbf{p}$ is orthogonal to $\mathbf{v}$.

If $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$ is an orthogonal set of vectors then

$$\mathbf{p} = \frac{\langle \mathbf{x}, \mathbf{v}_1 \rangle}{\langle \mathbf{v}_1, \mathbf{v}_1 \rangle} \mathbf{v}_1 + \frac{\langle \mathbf{x}, \mathbf{v}_2 \rangle}{\langle \mathbf{v}_2, \mathbf{v}_2 \rangle} \mathbf{v}_2 + \cdots + \frac{\langle \mathbf{x}, \mathbf{v}_n \rangle}{\langle \mathbf{v}_n, \mathbf{v}_n \rangle} \mathbf{v}_n$$

is the **orthogonal projection** of the vector $\mathbf{x}$ onto the subspace spanned by $\mathbf{v}_1, \ldots, \mathbf{v}_n$. That is, the remainder $\mathbf{o} = \mathbf{x} - \mathbf{p}$ is orthogonal to $\mathbf{v}_1, \ldots, \mathbf{v}_n$.



x
o
span of v_1, v_2
p
v_2
v_1

Orthogonalization

Let V be a vector space with an inner product. Suppose $\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_n$ is a basis for V. Let

$$\mathbf{v}_1 = \mathbf{x}_1,$$

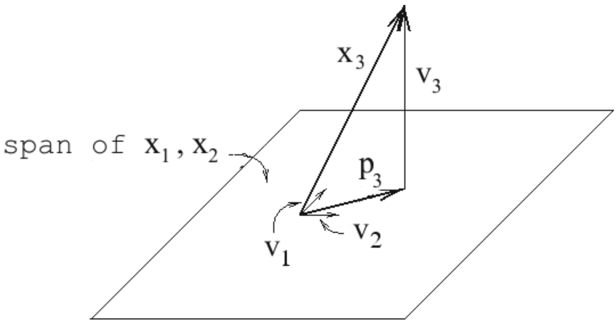
$$\mathbf{v}_2 = \mathbf{x}_2 - \frac{\langle \mathbf{x}_2, \mathbf{v}_1 \rangle}{\langle \mathbf{v}_1, \mathbf{v}_1 \rangle} \mathbf{v}_1,$$

$$\mathbf{v}_3 = \mathbf{x}_3 - \frac{\langle \mathbf{x}_3, \mathbf{v}_1 \rangle}{\langle \mathbf{v}_1, \mathbf{v}_1 \rangle} \mathbf{v}_1 - \frac{\langle \mathbf{x}_3, \mathbf{v}_2 \rangle}{\langle \mathbf{v}_2, \mathbf{v}_2 \rangle} \mathbf{v}_2,$$

....

$$\mathbf{v}_n = \mathbf{x}_n - \frac{\langle \mathbf{x}_n, \mathbf{v}_1 \rangle}{\langle \mathbf{v}_1, \mathbf{v}_1 \rangle} \mathbf{v}_1 - \cdots - \frac{\langle \mathbf{x}_n, \mathbf{v}_{n-1} \rangle}{\langle \mathbf{v}_{n-1}, \mathbf{v}_{n-1} \rangle} \mathbf{v}_{n-1}.$$

Then $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n$ is an orthogonal basis for V. The orthogonalization of a basis as described above is called the **Gram-Schmidt process**.



$$\mathbf{p}_3 = \frac{\langle \mathbf{x}_3, \mathbf{v}_1 \rangle}{\langle \mathbf{v}_1, \mathbf{v}_1 \rangle}\mathbf{v}_1 + \frac{\langle \mathbf{x}_3, \mathbf{v}_2 \rangle}{\langle \mathbf{v}_2, \mathbf{v}_2 \rangle}\mathbf{v}_2$$

Normalization

Let V be a vector space with an inner product. Suppose $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$ is an orthogonal basis for V.

Let $\mathbf{w}_1 = \dfrac{\mathbf{v}_1}{\|\mathbf{v}_1\|}$, $\mathbf{w}_2 = \dfrac{\mathbf{v}_2}{\|\mathbf{v}_2\|}, \ldots, \mathbf{w}_n = \dfrac{\mathbf{v}_n}{\|\mathbf{v}_n\|}$.

Then $\mathbf{w}_1, \mathbf{w}_2, \ldots, \mathbf{w}_n$ is an orthonormal basis for V.

Theorem Any finite-dimensional vector space with an inner product has an orthonormal basis.

Remark. An infinite-dimensional vector space with an inner product may or may not have an orthonormal basis.

Problem. Let Π be the plane in $\mathbb{R}^3$ spanned by vectors $\mathbf{x}_1 = (1, 2, 2)$ and $\mathbf{x}_2 = (-1, 0, 2)$.

(i) Find an orthonormal basis for Π.

(ii) Extend it to an orthonormal basis for $\mathbb{R}^3$.

$\mathbf{x}_1, \mathbf{x}_2$ is a basis for the plane Π. We can extend it to a basis for $\mathbb{R}^3$ by adding one vector from the standard basis. For instance, vectors $\mathbf{x}_1$, $\mathbf{x}_2$, and $\mathbf{x}_3 = (0, 0, 1)$ form a basis for $\mathbb{R}^3$ because

$$\begin{vmatrix} 1 & 2 & 2 \\ -1 & 0 & 2 \\ 0 & 0 & 1 \end{vmatrix} = \begin{vmatrix} 1 & 2 \\ -1 & 0 \end{vmatrix} = 2 \neq 0.$$

Using the Gram-Schmidt process, we orthogonalize the basis $\mathbf{x}_1 = (1, 2, 2)$, $\mathbf{x}_2 = (-1, 0, 2)$, $\mathbf{x}_3 = (0, 0, 1)$:

$$\mathbf{v}_1 = \mathbf{x}_1 = (1, 2, 2),$$

$$\mathbf{v}_2 = \mathbf{x}_2 - \frac{\langle \mathbf{x}_2, \mathbf{v}_1 \rangle}{\langle \mathbf{v}_1, \mathbf{v}_1 \rangle}\mathbf{v}_1 = (-1, 0, 2) - \frac{3}{9}(1, 2, 2)$$

$$= (-4/3, -2/3, 4/3),$$

$$\mathbf{v}_3 = \mathbf{x}_3 - \frac{\langle \mathbf{x}_3, \mathbf{v}_1 \rangle}{\langle \mathbf{v}_1, \mathbf{v}_1 \rangle}\mathbf{v}_1 - \frac{\langle \mathbf{x}_3, \mathbf{v}_2 \rangle}{\langle \mathbf{v}_2, \mathbf{v}_2 \rangle}\mathbf{v}_2$$

$$= (0, 0, 1) - \frac{2}{9}(1, 2, 2) - \frac{4/3}{4}(-4/3, -2/3, 4/3)$$

$$= (2/9, -2/9, 1/9).$$

Now $\mathbf{v}_1 = (1, 2, 2)$, $\mathbf{v}_2 = (-4/3, -2/3, 4/3)$, $\mathbf{v}_3 = (2/9, -2/9, 1/9)$ is an orthogonal basis for $\mathbb{R}^3$ while $\mathbf{v}_1, \mathbf{v}_2$ is an orthogonal basis for Π. It remains to normalize these vectors.

$$\langle \mathbf{v}_1, \mathbf{v}_1 \rangle = 9 \implies \|\mathbf{v}_1\| = 3$$
$$\langle \mathbf{v}_2, \mathbf{v}_2 \rangle = 4 \implies \|\mathbf{v}_2\| = 2$$
$$\langle \mathbf{v}_3, \mathbf{v}_3 \rangle = 1/9 \implies \|\mathbf{v}_3\| = 1/3$$

$$\mathbf{w}_1 = \mathbf{v}_1/\|\mathbf{v}_1\| = (1/3, 2/3, 2/3) = \tfrac{1}{3}(1, 2, 2),$$
$$\mathbf{w}_2 = \mathbf{v}_2/\|\mathbf{v}_2\| = (-2/3, -1/3, 2/3) = \tfrac{1}{3}(-2, -1, 2),$$
$$\mathbf{w}_3 = \mathbf{v}_3/\|\mathbf{v}_3\| = (2/3, -2/3, 1/3) = \tfrac{1}{3}(2, -2, 1).$$

$\mathbf{w}_1, \mathbf{w}_2$ is an orthonormal basis for Π.
$\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3$ is an orthonormal basis for $\mathbb{R}^3$.