MATH 311-504

Topics in Applied Mathematics

Lecture 9:
Inverse matrix.

Identity matrix

Definition. The **identity matrix** (or **unit matrix**) is a diagonal matrix with all diagonal entries equal to 1. The $n\times n$ identity matrix is denoted I_n or simply I.

$$I_1 = (1), \quad I_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad I_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

In general, $I = \begin{pmatrix} 1 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 1 \end{pmatrix}.$

Theorem. Let A be an arbitrary $m\times n$ matrix. Then $I_m A = A I_n = A$.

Inverse matrix

Notation. $\mathcal{M}_n(\mathbb{R})$ denote the set of all $n \times n$ matrices with real entries.

Definition. Let $A \in \mathcal{M}_n(\mathbb{R})$. Suppose there exists an $n \times n$ matrix B such that

$$AB = BA = I_n.$$

Then the matrix A is called **invertible** and B is called the **inverse** of A (denoted A^{-1}).

$$\boxed{AA^{-1} = A^{-1}A = I}$$

Examples

$$A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}, \quad C = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}.$$

$$AB = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$

$$BA = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$

$$C^2 = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Thus $A^{-1} = B$, $B^{-1} = A$, and $C^{-1} = C$.

Example. $A = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$.

In the previous lecture it was shown that $A^2 - 3A + I = O$. Assume that the matrix A is invertible. Then

$$A^2 - 3A + I = O \implies A^{-1}(A^2 - 3A + I) = A^{-1}O$$
$$\implies A^{-1}AA - 3A^{-1}A + A^{-1}I = O$$
$$\implies A - 3I + A^{-1} = O \implies A^{-1} = 3I - A$$

The above argument suggests (but **does not prove**) that the matrix $B = 3I - A = \begin{pmatrix} 1 & -1 \\ -1 & 2 \end{pmatrix}$ is the inverse of A.

And, indeed, $AB = BA = (3I - A)A = 3A - A^2 = I$.

Basic properties of inverse matrices:

• If $B = A^{-1}$ then $A = B^{-1}$. In other words, if A is invertible, so is A^{-1}, and $A = (A^{-1})^{-1}$.

• The inverse matrix (if it exists) is unique. Moreover, if $AB = CA = I$ for some matrices $B, C \in \mathcal{M}_n(\mathbb{R})$ then $B = C = A^{-1}$.

Indeed, $B = IB = (CA)B = C(AB) = CI = C$.

• If matrices $A, B \in \mathcal{M}_n(\mathbb{R})$ are invertible, so is AB, and $(AB)^{-1} = B^{-1}A^{-1}$.

$$(B^{-1}A^{-1})(AB) = B^{-1}(A^{-1}A)B = B^{-1}IB = B^{-1}B = I,$$
$$(AB)(B^{-1}A^{-1}) = A(BB^{-1})A^{-1} = AIA^{-1} = AA^{-1} = I.$$

• Similarly, $(A_1A_2\dots A_k)^{-1} = A_k^{-1}\dots A_2^{-1}A_1^{-1}$.

Other examples

$$D = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad E = \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}.$$

$$D^2 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.$$

It follows that D is not invertible as otherwise

$$D^2 = O \implies D^{-1}D^2 = D^{-1}O \implies D = O.$$

$$E^2 = \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} = \begin{pmatrix} 2 & -2 \\ -2 & 2 \end{pmatrix} = 2E.$$

It follows that E is not invertible as otherwise

$$E^2 = 2E \implies E^2E^{-1} = 2EE^{-1} \implies E = 2I.$$

Theorem Suppose that D and E are n-by-n matrices such that $\boxed{DE = O.}$ Then exactly one of the following is true:

(i) D is invertible, $E = O$;
(ii) $D = O$, E is invertible;
(iii) neither D nor E is invertible.

Proof: If D is invertible then

$$DE = O \implies D^{-1}DE = D^{-1}O \implies E = O.$$

If E is invertible then

$$DE = O \implies DEE^{-1} = OE^{-1} \implies D = O.$$

It remains to notice that the zero matrix is not invertible.

Inverting diagonal matrices

Theorem A diagonal matrix $D = \mathrm{diag}(d_1, \ldots, d_n)$ is invertible if and only if all diagonal entries are nonzero: $d_i \neq 0$ for $1 \leq i \leq n$.

If D is invertible then $D^{-1} = \mathrm{diag}(d_1^{-1}, \ldots, d_n^{-1})$.

$$\begin{pmatrix} d_1 & 0 & \ldots & 0 \\ 0 & d_2 & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & d_n \end{pmatrix}^{-1} = \begin{pmatrix} d_1^{-1} & 0 & \ldots & 0 \\ 0 & d_2^{-1} & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & d_n^{-1} \end{pmatrix}$$

Inverting diagonal matrices

Theorem A diagonal matrix $D = \mathrm{diag}(d_1, \ldots, d_n)$ is invertible if and only if all diagonal entries are nonzero: $d_i \neq 0$ for $1 \leq i \leq n$.

If D is invertible then $D^{-1} = \mathrm{diag}(d_1^{-1}, \ldots, d_n^{-1})$.

Proof: If all $d_i \neq 0$ then, clearly,

$$\mathrm{diag}(d_1, \ldots, d_n)\,\mathrm{diag}(d_1^{-1}, \ldots, d_n^{-1}) = \mathrm{diag}(1, \ldots, 1) = I,$$
$$\mathrm{diag}(d_1^{-1}, \ldots, d_n^{-1})\,\mathrm{diag}(d_1, \ldots, d_n) = \mathrm{diag}(1, \ldots, 1) = I.$$

Now suppose that $d_i = 0$ for some i. Then for any $n \times n$ matrix B the ith row of the matrix DB is a zero row. Hence $DB \neq I$.

Inverting 2-by-2 matrices

Definition. The **determinant** of a 2×2 matrix $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is $\det A = ad - bc$.

Theorem A matrix $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is invertible if and only if $\det A \neq 0$.

If $\det A \neq 0$ then

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} = \frac{1}{ad - bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}.$$

Theorem A matrix $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is invertible if and only if $\det A \neq 0$. If $\det A \neq 0$ then

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} = \frac{1}{ad - bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}.$$

Proof: Let $B = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$. Then

$$AB = BA = \begin{pmatrix} ad-bc & 0 \\ 0 & ad-bc \end{pmatrix} = (ad - bc)I_2.$$

In the case $\det A \neq 0$, we have $A^{-1} = (\det A)^{-1}B$.
In the case $\det A = 0$, the matrices A and B are not invertible because $A = O \iff B = O$.

Fundamental results on inverse matrices

Theorem 1 Given a square matrix A, the following are equivalent:

(i) A is invertible;
(ii) $\mathbf{x} = \mathbf{0}$ is the only solution of the matrix equation $A\mathbf{x} = \mathbf{0}$;
(iii) the row echelon form of A has no zero rows;
(iv) the reduced row echelon form of A is the identity matrix.

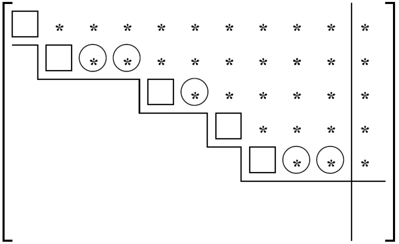
Theorem 2 Suppose that a sequence of elementary row operations converts a matrix A into the identity matrix.

Then the same sequence of operations converts the identity matrix into the inverse matrix A^{-1}.

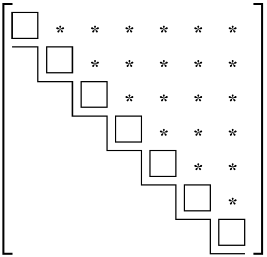
Theorem 3 For any $n \times n$ matrices A and B,

$$BA = I \iff AB = I.$$

Row echelon form of a square matrix:



noninvertible case



invertible case