MATH 311

Topics in Applied Mathematics I

Lecture 18:
Change of basis (continued).
Linear transformations.

Basis and coordinates

If $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ is a basis for a vector space V, then any vector $\mathbf{v} \in V$ has a unique representation

$$\mathbf{v} = x_1\mathbf{v}_1 + x_2\mathbf{v}_2 + \cdots + x_n\mathbf{v}_n,$$

where $x_i \in \mathbb{R}$. The coefficients $x_1, x_2, \ldots, x_n$ are called the **coordinates** of $\mathbf{v}$ with respect to the ordered basis $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$.

The mapping

$$\textit{vector } \mathbf{v} \mapsto \textit{its coordinates } (x_1, x_2, \ldots, x_n)$$

is a one-to-one correspondence between V and $\mathbb{R}^n$. This correspondence respects linear operations in V and in $\mathbb{R}^n$.

Change of coordinates

Let V be a vector space of dimension n.

Let $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$ be a basis for V and $g_1 : V \to \mathbb{R}^n$ be the coordinate mapping corresponding to this basis.

Let $\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_n$ be another basis for V and $g_2 : V \to \mathbb{R}^n$ be the coordinate mapping corresponding to this basis.

$$\begin{array}{ccccc} & & V & & \\ & \overset{g_1}{\swarrow} & & \overset{g_2}{\searrow} & \\ \mathbb{R}^n & & \longrightarrow & & \mathbb{R}^n \end{array}$$

The composition $g_2 \circ g_1^{-1}$ is a transformation of $\mathbb{R}^n$.
It has the form $\mathbf{x} \mapsto U\mathbf{x}$, where U is an $n \times n$ matrix.

U is called the **transition matrix** from $\mathbf{v}_1, \mathbf{v}_2 \ldots, \mathbf{v}_n$ to $\mathbf{u}_1, \mathbf{u}_2 \ldots, \mathbf{u}_n$. Columns of U are coordinates of the vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$ with respect to the basis $\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_n$.

Problem. Find the transition matrix from the basis $\mathbf{v}_1 = (1,2,3)$, $\mathbf{v}_2 = (1,0,1)$, $\mathbf{v}_3 = (1,2,1)$ to the basis $\mathbf{u}_1 = (1,1,0)$, $\mathbf{u}_2 = (0,1,1)$, $\mathbf{u}_3 = (1,1,1)$.

It is convenient to make a two-step transition: first from $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ to $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$, and then from $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$ to $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3$.

Let U_1 be the transition matrix from $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ to $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$ and U_2 be the transition matrix from $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3$ to $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$:

$$U_1 = \begin{pmatrix} 1 & 1 & 1 \\ 2 & 0 & 2 \\ 3 & 1 & 1 \end{pmatrix}, \qquad U_2 = \begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{pmatrix}.$$

Basis $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3 \implies$ coordinates $\mathbf{x}$

Basis $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3 \implies$ coordinates $U_1\mathbf{x}$

Basis $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3 \implies$ coordinates $U_2^{-1}(U_1\mathbf{x}) = (U_2^{-1}U_1)\mathbf{x}$

Thus the transition matrix from $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ to $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3$ is $U_2^{-1}U_1$.

$$U_2^{-1}U_1 = \begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{pmatrix}^{-1} \begin{pmatrix} 1 & 1 & 1 \\ 2 & 0 & 2 \\ 3 & 1 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 1 & -1 \\ -1 & 1 & 0 \\ 1 & -1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 & 1 \\ 2 & 0 & 2 \\ 3 & 1 & 1 \end{pmatrix} = \begin{pmatrix} -1 & -1 & 1 \\ 1 & -1 & 1 \\ 2 & 2 & 0 \end{pmatrix}.$$

Linear mapping = linear transformation = linear function

Definition. Given vector spaces V_1 and V_2, a mapping $L : V_1 \to V_2$ is **linear** if

$$\boxed{L(\mathbf{x} + \mathbf{y}) = L(\mathbf{x}) + L(\mathbf{y}),}$$

$$\boxed{L(r\mathbf{x}) = rL(\mathbf{x})}$$

for any $\mathbf{x}, \mathbf{y} \in V_1$ and $r \in \mathbb{R}$.

A linear mapping $\ell : V \to \mathbb{R}$ is called a **linear functional** on V.

If $V_1 = V_2$ (or if both V_1 and V_2 are functional spaces) then a linear mapping $L : V_1 \to V_2$ is called a **linear operator**.

Linear mapping = linear transformation = linear function

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for any $\mathbf{x}, \mathbf{y} \in V_1$ and $r \in \mathbb{R}$.

Remark. A function $f : \mathbb{R} \to \mathbb{R}$ given by $f(x) = ax + b$ is a linear transformation of the vector space $\mathbb{R}$ if and only if $b = 0$.

Basic properties of linear transformations

Let $L : V_1 \to V_2$ be a linear mapping.

- $L(r_1\mathbf{v}_1 + \cdots + r_k\mathbf{v}_k) = r_1 L(\mathbf{v}_1) + \cdots + r_k L(\mathbf{v}_k)$

for all $k \geq 1$, $\mathbf{v}_1, \dots, \mathbf{v}_k \in V_1$, and $r_1, \dots, r_k \in \mathbb{R}$.

$L(r_1\mathbf{v}_1 + r_2\mathbf{v}_2) = L(r_1\mathbf{v}_1) + L(r_2\mathbf{v}_2) = r_1 L(\mathbf{v}_1) + r_2 L(\mathbf{v}_2)$,
$L(r_1\mathbf{v}_1 + r_2\mathbf{v}_2 + r_3\mathbf{v}_3) = L(r_1\mathbf{v}_1 + r_2\mathbf{v}_2) + L(r_3\mathbf{v}_3) =$
$= r_1 L(\mathbf{v}_1) + r_2 L(\mathbf{v}_2) + r_3 L(\mathbf{v}_3)$, and so on.

- $L(\mathbf{0}_1) = \mathbf{0}_2$, where $\mathbf{0}_1$ and $\mathbf{0}_2$ are zero vectors in V_1 and V_2, respectively.

$L(\mathbf{0}_1) = L(0\mathbf{0}_1) = 0L(\mathbf{0}_1) = \mathbf{0}_2$.

- $L(-\mathbf{v}) = -L(\mathbf{v})$ for any $\mathbf{v} \in V_1$.

$L(-\mathbf{v}) = L((-1)\mathbf{v}) = (-1)L(\mathbf{v}) = -L(\mathbf{v})$.

Examples of linear mappings

- *Scaling* $L : V \to V$, $L(\mathbf{v}) = s\mathbf{v}$, where $s \in \mathbb{R}$.

$L(\mathbf{x}+\mathbf{y}) = s(\mathbf{x}+\mathbf{y}) = s\mathbf{x} + s\mathbf{y} = L(\mathbf{x}) + L(\mathbf{y})$,
$L(r\mathbf{x}) = s(r\mathbf{x}) = r(s\mathbf{x}) = rL(\mathbf{x})$.

- *Dot product with a fixed vector*

$\ell : \mathbb{R}^n \to \mathbb{R}$, $\ell(\mathbf{v}) = \mathbf{v} \cdot \mathbf{v}_0$, where $\mathbf{v}_0 \in \mathbb{R}^n$.

$\ell(\mathbf{x}+\mathbf{y}) = (\mathbf{x}+\mathbf{y}) \cdot \mathbf{v}_0 = \mathbf{x} \cdot \mathbf{v}_0 + \mathbf{y} \cdot \mathbf{v}_0 = \ell(\mathbf{x}) + \ell(\mathbf{y})$,
$\ell(r\mathbf{x}) = (r\mathbf{x}) \cdot \mathbf{v}_0 = r(\mathbf{x} \cdot \mathbf{v}_0) = r\ell(\mathbf{x})$.

- *Cross product with a fixed vector*

$L : \mathbb{R}^3 \to \mathbb{R}^3$, $L(\mathbf{v}) = \mathbf{v} \times \mathbf{v}_0$, where $\mathbf{v}_0 \in \mathbb{R}^3$.

- *Multiplication by a fixed matrix*

$L : \mathbb{R}^n \to \mathbb{R}^m$, $L(\mathbf{v}) = A\mathbf{v}$, where A is an $m \times n$ matrix and all vectors are column vectors.

Linear mappings of functional vector spaces

- *Evaluation at a fixed point*

$\ell : F(\mathbb{R}) \to \mathbb{R}, \ \ell(f) = f(a), \text{ where } a \in \mathbb{R}.$

- *Multiplication by a fixed function*

$L : F(\mathbb{R}) \to F(\mathbb{R}), \ L(f) = gf, \text{ where } g \in F(\mathbb{R}).$

- *Differentiation* $D : C^1(\mathbb{R}) \to C(\mathbb{R}), \ L(f) = f'.$

$D(f+g) = (f+g)' = f' + g' = D(f) + D(g),$
$D(rf) = (rf)' = rf' = rD(f).$

- *Integration over a finite interval*

$\ell : C(\mathbb{R}) \to \mathbb{R}, \ \ell(f) = \int_a^b f(x)\,dx, \text{ where}$
$a, b \in \mathbb{R}, \ a < b.$

More properties of linear mappings

- If a linear mapping $L : V \to W$ is invertible then the inverse mapping $L^{-1} : W \to V$ is also linear.

- If $L : V \to W$ and $M : W \to X$ are linear mappings then the composition $M \circ L : V \to X$ is also linear.

- If $L_1 : V \to W$ and $L_2 : V \to W$ are linear mappings then the sum $L_1 + L_2$ is also linear.

Linear differential operators

- an ordinary differential operator

$$L : C^\infty(\mathbb{R}) \to C^\infty(\mathbb{R}), \qquad L = g_0 \frac{d^2}{dx^2} + g_1 \frac{d}{dx} + g_2,$$

where g_0, g_1, g_2 are smooth functions on $\mathbb{R}$.
That is, $L(f) = g_0 f'' + g_1 f' + g_2 f$.

- Laplace's operator $\Delta : C^\infty(\mathbb{R}^2) \to C^\infty(\mathbb{R}^2)$,

$$\Delta f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2}$$

(a.k.a. the Laplacian; also denoted by ∇^2).

Linear integral operators

- anti-derivative

$$L : C[a,b] \to C^1[a,b], \quad (Lf)(x) = \int_a^x f(y)\,dy.$$

- Hilbert-Schmidt operator

$$L : C[a,b] \to C[c,d], \quad (Lf)(x) = \int_a^b K(x,y)f(y)\,dy,$$

where $K \in C([c,d] \times [a,b])$.

- Laplace transform

$$\mathcal{L} : BC(0,\infty) \to C(0,\infty), \quad (\mathcal{L}f)(x) = \int_0^\infty e^{-xy} f(y)\,dy.$$

Examples. $\mathcal{M}_{m,n}(\mathbb{R})$: the space of $m\times n$ matrices.

- $\alpha : \mathcal{M}_{m,n}(\mathbb{R}) \to \mathcal{M}_{n,m}(\mathbb{R}),\ \ \alpha(A) = A^T.$

$\alpha(A+B) = \alpha(A) + \alpha(B) \iff (A+B)^T = A^T + B^T.$
$\alpha(rA) = r\,\alpha(A) \iff (rA)^T = rA^T.$
Hence α is linear.

- $\beta : \mathcal{M}_{2,2}(\mathbb{R}) \to \mathbb{R},\ \ \beta(A) = \det A.$

Let $A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ and $B = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$.

Then $A + B = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$.

We have $\det(A) = \det(B) = 0$ while $\det(A+B) = 1$.
Hence $\beta(A+B) \neq \beta(A) + \beta(B)$ so that β is not linear.

Range and kernel

Let V, W be vector spaces and $L : V \to W$ be a linear mapping.

Definition. The **range** (or **image**) of L is the set of all vectors $\mathbf{w} \in W$ such that $\mathbf{w} = L(\mathbf{v})$ for some $\mathbf{v} \in V$. The range of L is denoted $L(V)$.

The **kernel** of L, denoted $\ker L$, is the set of all vectors $\mathbf{v} \in V$ such that $L(\mathbf{v}) = \mathbf{0}$.

Theorem **(i)** The range of L is a subspace of W.
(ii) The kernel of L is a subspace of V.