MATH 311

Topics in Applied Mathematics I

Lecture 31:
Differentiation in vector spaces. Gradient, divergence, and curl.

The derivative

Definition. A real function f is said to be **differentiable** at a point $a \in \mathbb{R}$ if it is defined on an open interval containing a and the limit

$$\lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

exists. The limit is denoted $f'(a)$ and called the **derivative** of f at a. An equivalent condition is

$$f(a+h) = f(a) + f'(a)h + r(h), \text{ where } \lim_{h \to 0} r(h)/h = 0.$$

If a function f is differentiable at a point a, then it is continuous at a.

Suppose that a function f is defined and differentiable on an interval I. Then the derivative of f can be regarded as a function on I.

Convergence in normed vector spaces

Suppose V is a vector space endowed with a norm $\|\cdot\|$. The norm gives rise to a distance function $\operatorname{dist}(\mathbf{x}, \mathbf{y}) = \|\mathbf{x} - \mathbf{y}\|$.

Definition. We say that a sequence of vectors $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \dots$ converges to a vector $\mathbf{u}$ in the normed vector space V if $\|\mathbf{v}_k - \mathbf{u}\| \to 0$ as $k \to \infty$.

In the case $V = \mathbb{R}^n$, a sequence of vectors converges with respect to a norm if and only if it converges in each coordinate. In the case $V = \mathcal{M}_{m,n}(\mathbb{R})$, a sequence of matrices converges with respect to a norm if and only if it converges in each entry.

Similarly, in the case $\dim V < \infty$ we can choose a finite basis $\mathbf{w}_1, \mathbf{w}_2, \dots, \mathbf{w}_n$. Any vector $\mathbf{v} \in V$ can be expanded into a linear combination $\mathbf{v} = x_1\mathbf{w}_1 + x_2\mathbf{w}_2 + \cdots + x_n\mathbf{w}_n$. Then a sequence of vectors converges with respect to a norm if and only if each of the coordinates x_i converges.

Vector-valued functions

Suppose V is a vector space endowed with a norm $\|\cdot\|$.

Definition. We say that a function $\mathbf{v} : X \to V$ defined on a set $X \subset \mathbb{R}$ converges to a limit $\mathbf{u} \in V$ at a point $a \in \mathbb{R}$ if $\|\mathbf{v}(x) - \mathbf{u}\| \to 0$ as $x \to a$.

Further, we say that the function $\mathbf{v}$ is continuous at a point $c \in X$ if $\mathbf{v}(c) = \lim\limits_{x \to c} \mathbf{v}(x)$.

Finally, the function $\mathbf{v}$ is said to be differentiable at a point $a \in \mathbb{R}$ if it is defined on an open interval containing a and the limit

$$\lim_{h \to 0} \frac{1}{h}\big(f(a+h) - f(a)\big)$$

exists. The limit is denoted $\mathbf{v}'(a)$ and called the derivative of $\mathbf{v}$ at a.

Differentiability theorems

Sum Rule If functions $\mathbf{v} : X \to V$ and $\mathbf{w} : X \to V$ are differentiable at a point $a \in \mathbb{R}$, then the sum $\mathbf{v} + \mathbf{w}$ is also differentiable at a. Moreover, $(\mathbf{v} + \mathbf{w})'(a) = \mathbf{v}'(a) + \mathbf{w}'(a)$.

Homogeneous Rule If a function $\mathbf{v} : X \to V$ is differentiable at a point $a \in \mathbb{R}$, then for any $r \in \mathbb{R}$ the scalar multiple $r\mathbf{v}$ is also differentiable at a. Moreover, $(r\mathbf{v})'(a) = r\mathbf{v}'(a)$.

Difference Rule If functions $\mathbf{v} : X \to V$ and $\mathbf{w} : X \to V$ are differentiable at a point $a \in \mathbb{R}$, then the difference $\mathbf{v} - \mathbf{w}$ is also differentiable at a. Moreover, $(\mathbf{v} - \mathbf{w})'(a) = \mathbf{v}'(a) - \mathbf{w}'(a)$.

Differentiability theorems

Product Rule #1 If functions $f : X \to \mathbb{R}$ and $\mathbf{v} : X \to V$ are differentiable at a point $a \in \mathbb{R}$, then the scalar multiple $f\mathbf{v}$ is also differentiable at a. Moreover, $(f\mathbf{v})'(a) = f'(a)\mathbf{v}(a) + f(a)\mathbf{v}'(a)$.

Product Rule #2 Assume that the norm on V is induced by an inner product $\langle \cdot, \cdot \rangle$. If functions $\mathbf{v} : X \to V$ and $\mathbf{w} : X \to V$ are differentiable at a point $a \in \mathbb{R}$, then the inner product $\langle \mathbf{v}, \mathbf{w} \rangle$ is also differentiable at a. Moreover, $(\langle \mathbf{v}, \mathbf{w} \rangle)'(a) = \langle \mathbf{v}'(a), \mathbf{w}(a) \rangle + \langle \mathbf{v}(a), \mathbf{w}'(a) \rangle$.

Chain Rule If a function $f : X \to \mathbb{R}$ is differentiable at a point $a \in \mathbb{R}$ and a function $\mathbf{v} : Y \to V$ is differentiable at $f(a)$, then the composition $\mathbf{v} \circ f$ is differentiable at a. Moreover, $(\mathbf{v} \circ f)'(a) = f'(a)\mathbf{v}'(f(a))$.

Partial derivative

Consider a function $f : X \to V$ that is defined in a domain $X \subset \mathbb{R}^n$ and takes values in a normed vector space V. The function f depends on n real variables: $f = f(x_1, x_2, \dots, x_n)$.

Let us select a point $\mathbf{a} = (a_1, a_2, \dots, a_n) \in X$ and a variable x_i. Now we go to the point $\mathbf{a}$ and fix all variables except x_i. That is, we introduce a function of one variable

$$\phi(x) = f(a_1, \dots, a_{i-1}, x, a_{i+1}, \dots, a_n).$$

If the function ϕ is differentiable at a_i, then the derivative $\phi'(a_i)$ is called the **partial derivative** of f at the point $\mathbf{a}$ with respect to the variable x_i.

Notation: $\dfrac{\partial f}{\partial x_i}(\mathbf{a}),\ \dfrac{\partial}{\partial x_i} f(\mathbf{a}),\ (D_{x_i} f)(\mathbf{a}).$

Directional derivative

Consider a function $f : X \to V$ that is defined on a subset $X \subset W$ of a vector space W and takes values in a normed vector space V. For every point $\mathbf{a} \in X$ and vector $\mathbf{v} \in W$ we introduce a function of real variable $\phi(t) = f(\mathbf{a} + t\mathbf{v})$. If the function ϕ is differentiable at 0, then the derivative $\phi'(0)$ is called the **directional derivative** of f at the point $\mathbf{a}$ along the vector $\mathbf{v}$. Notation: $(D_{\mathbf{v}}f)(\mathbf{a})$.

The partial derivative is a particular case of the directional derivative, when $W = \mathbb{R}^n$ and $\mathbf{v}$ is from the standard basis.

Homogeneity $(D_{r\mathbf{v}}f)(\mathbf{a}) = r(D_{\mathbf{v}}f)(\mathbf{a})$ for all $r \in \mathbb{R}$ whenever $(D_{\mathbf{v}}f)(\mathbf{a})$ exists.

Linearity Suppose W is a normed vector space, $(D_{\mathbf{v}}f)(\mathbf{a})$ exists for all $\mathbf{v}$ and depends continuously on $\mathbf{a}$. Then $\mathbf{v} \mapsto (D_{\mathbf{v}}f)(\mathbf{a})$ is a linear transformation.

The differential

Suppose V and W are normed vector spaces and consider a function $F : X \to V$, where $X \subset W$.

Definition. We say that the function F is **differentiable** at a point $\mathbf{a} \in X$ if it is defined in a neighborhood of **a** and there exists a continuous linear transformation $L : W \to V$ such that $F(\mathbf{a}+\mathbf{v}) = F(\mathbf{a}) + L(\mathbf{v}) + R(\mathbf{v})$, where $\|R(\mathbf{v})\|/\|\mathbf{v}\| \to 0$ as $\|\mathbf{v}\| \to 0$. The transformation L is called the **differential** of F at **a** and denoted $(DF)(\mathbf{a})$.

Theorem If a function F is differentiable at a point **a**, then the directional derivatives $(D_{\mathbf{v}}F)(\mathbf{a})$ exist for all **v** and $(D_{\mathbf{v}}F)(\mathbf{a}) = (DF)(\mathbf{a})[\mathbf{v}]$.

Fermat's Theorem If a real-valued function F is differentiable at a point **a** of local extremum, then the differential $(DF)(\mathbf{a})$ is identically zero.

Gradient, divergence, and curl

Gradient of a scalar field $f = f(x_1, x_2 \ldots, x_n)$ is

$$\operatorname{grad} f = \Big(\frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_2}, \ldots, \frac{\partial f}{\partial x_n}\Big).$$

Divergence of a vector field $\mathbf{F} = (F_1, F_2, \ldots, F_n)$ is

$$\operatorname{div} \mathbf{F} = \frac{\partial F_1}{\partial x_1} + \frac{\partial F_2}{\partial x_2} + \cdots + \frac{\partial F_n}{\partial x_n}.$$

Curl of a vector field $\mathbf{F} = (F_1, F_2, F_3)$ is

$$\operatorname{curl} \mathbf{F} = \Big(\frac{\partial F_3}{\partial x_2} - \frac{\partial F_2}{\partial x_3},\ \frac{\partial F_1}{\partial x_3} - \frac{\partial F_3}{\partial x_1},\ \frac{\partial F_2}{\partial x_1} - \frac{\partial F_1}{\partial x_2}\Big).$$

Informally, $\operatorname{curl} \mathbf{F} = \begin{vmatrix} \mathbf{e}_1 & \mathbf{e}_2 & \mathbf{e}_3 \\ \frac{\partial}{\partial x_1} & \frac{\partial}{\partial x_2} & \frac{\partial}{\partial x_3} \\ F_1 & F_2 & F_3 \end{vmatrix}.$

Del notation

Gradient, divergence, and curl can be denoted in a compact way using the del (a.k.a. nabla a.k.a. atled) "operator"

$$\nabla = \Big(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \dots, \frac{\partial}{\partial x_n}\Big).$$

Namely, $\operatorname{grad} f = \nabla f$, $\operatorname{div} \mathbf{F} = \nabla \cdot \mathbf{F}$, $\operatorname{curl} \mathbf{F} = \nabla \times \mathbf{F}$.

Theorem 1 $\operatorname{div}(\operatorname{curl} \mathbf{F}) = 0$ wherever the vector field $\mathbf{F}$ is twice continuously differentiable.

Theorem 2 $\operatorname{curl}(\operatorname{grad} f) = \mathbf{0}$ wherever the scalar field f is twice continuously differentiable.

In the del notation, $\nabla \cdot (\nabla \times \mathbf{F}) = 0$ and $\nabla \times (\nabla f) = \mathbf{0}$.