MATH 311

Topics in Applied Mathematics I

Lecture 33:
Multiple integrals.
Line integrals.

Riemann sums in two dimensions

Consider a closed coordinate rectangle
$R = [a, b] \times [c, d] \subset \mathbb{R}^2$.

Definition. A **Riemann sum** of a function $f : R \to \mathbb{R}$ with respect to a partition $P = \{D_1, D_2, \dots, D_n\}$ of R generated by samples $t_j \in D_j$ is a sum

$$\mathcal{S}(f, P, t_j) = \sum\nolimits_{j=1}^{n} f(t_j)\, \mathrm{area}(D_j).$$

The norm of the partition P is $\|P\| = \max_{1 \le j \le n} \mathrm{diam}(D_j)$.

Definition. The Riemann sums $\mathcal{S}(f, P, t_j)$ **converge** to a limit $I(f)$ as the norm $\|P\| \to 0$ if for every $\varepsilon > 0$ there exists $\delta > 0$ such that $\|P\| < \delta$ implies $|\mathcal{S}(f, P, t_j) - I(f)| < \varepsilon$ for any partition P and choice of samples t_j.

If this is the case, then the function f is called **integrable** on R and the limit $I(f)$ is called the **integral** of f over R.

Double integral

Closed coordinate rectangle $R = [a, b] \times [c, d]$
$= \{(x, y) \in \mathbb{R}^2 \mid a \le x \le b, \;\; c \le y \le d\}$.

Notation: $\displaystyle\iint_R f\, dA$ or $\displaystyle\iint_R f(x, y)\, dx\, dy$.

Theorem 1 If f is continuous on the closed rectangle R, then f is integrable.

Theorem 2 A function $f : R \to \mathbb{R}$ is Riemann integrable on the rectangle R if and only if f is bounded on R and continuous almost everywhere on R (that is, the set of discontinuities of f has zero area).

Fubini's Theorem

Fubini's Theorem allows us to reduce a multiple integral to a repeated one-dimensional integral.

Theorem If a function f is integrable on $R = [a, b] \times [c, d]$, then

$$\iint_R f\, dA = \int_a^b \Big(\int_c^d f(x, y)\, dy \Big)\, dx$$
$$= \int_c^d \Big(\int_a^b f(x, y)\, dx \Big)\, dy.$$

In particular, this implies that we can change the order of integration in a repeated integral.

Integrals over general domains

Suppose $f : D \to \mathbb{R}$ is a function defined on a (Jordan) measurable set $D \subset \mathbb{R}^2$. Since D is bounded, it is contained in a rectangle R. To define the integral of f over D, we extend the function f to a function on R:

$$f^{\rm ext}(x, y) = \begin{cases} f(x, y) & \text{if } (x, y) \in D, \\ 0 & \text{if } (x, y) \notin D. \end{cases}$$

Definition. $\iint_D f \, dA$ is defined to be $\iint_R f^{\rm ext} \, dA$.

In particular, $\mathrm{area}(D) = \iint_D 1 \, dA$.

Integration as a linear operation

Theorem 1 If functions f, g are integrable on a set $D \subset \mathbb{R}^2$, then the sum $f + g$ is also integrable on D and

$$\iint_D (f+g)\, dA = \iint_D f\, dA + \iint_D g\, dA.$$

Theorem 2 If a function f is integrable on a set $D \subset \mathbb{R}^2$, then for each $\alpha \in \mathbb{R}$ the scalar multiple αf is also integrable on D and

$$\iint_D \alpha f\, dA = \alpha \iint_D f\, dA.$$

Comparison theorems for integrals

Theorem 1 If functions f, g are integrable on a set $D \subset \mathbb{R}^2$, and $f(x,y) \leq g(x,y)$ for all $(x,y) \in D$, then

$$\iint_D f\,dA \leq \iint_D g\,dA.$$

Theorem 2 If f is integrable on a set $D \subset \mathbb{R}^2$, then the function $|f|$ is also integrable on D and

$$\left|\iint_D f\,dA\right| \leq \iint_D |f|\,dA.$$

More properties of integrals

Theorem If a function $f : D \to \mathbb{R}$ is integrable on the set $D \subset \mathbb{R}^2$ and $f(D) \subset [a, b]$, then for each continuous function $g : [a, b] \to \mathbb{R}$ the composition $g \circ f$ is also integrable on D.

Theorem If a function f is integrable on sets $D_1, D_2 \subset \mathbb{R}^2$, then it is integrable on their union $D_1 \cup D_2$. Moreover, if the sets D_1 and D_2 are disjoint up to a set of zero area, then

$$\iint_{D_1 \cup D_2} f \, dA = \iint_{D_1} f \, dA + \iint_{D_2} f \, dA.$$

Change of variables in a double integral

Theorem Let $D \subset \mathbb{R}^2$ be a measurable domain and f be an integrable function on D. If $\mathbf{T} = (u, v)$ is a smooth coordinate mapping such that $\mathbf{T}^{-1}$ is defined on D, Then

$$\iint_D f(u, v)\, du\, dv$$

$$= \iint_{\mathbf{T}^{-1}(D)} f\left(u(x, y), v(x, y)\right) \left|\det \frac{\partial(u, v)}{\partial(x, y)}\right| \, dx\, dy.$$

In particular, the integral in the right-hand side is well-defined.

Triple integral

To integrate in $\mathbb{R}^3$, volumes are used instead of areas in $\mathbb{R}^2$. Instead of coordinate rectangles, basic sets are coordinate boxes (or bricks) $B = [a_1, b_1] \times [a_2, b_2] \times [a_3, b_3] \subset \mathbb{R}^3$. Then we can define an integral of a function f over a measurable set $D \subset \mathbb{R}^3$.

Notation: $\iiint_D f \, dV$ or $\iiint_D f(x, y, z) \, dx \, dy \, dz$.

The properties of triple integrals are completely analogous to those of double integrals. In particular, Fubini's Theorem is formulated as follows.

Theorem If a function f is integrable on a brick $B = [a_1, b_1] \times [a_2, b_2] \times [a_3, b_3] \subset \mathbb{R}^3$, then

$$\iiint_B f \, dV = \int_{a_1}^{b_1} \left(\int_{a_2}^{b_2} \left(\int_{a_3}^{b_3} f(x, y, z) \, dz \right) dy \right) dx.$$

Path

Definition. A **path** in $\mathbb{R}^n$ is a continuous function $\mathbf{x} : [a, b] \to \mathbb{R}^n$.

Paths provide parametrizations for curves.

Length of the path $\mathbf{x}$ is defined as $L = \sup_P \sum_{j=1}^{k} \|\mathbf{x}(t_j) - \mathbf{x}(t_{j-1})\|$ over all partitions $P = \{t_0, t_1, \ldots, t_k\}$ of the interval $[a, b]$.

Theorem The length of a smooth path $\mathbf{x} : [a, b] \to \mathbb{R}^n$ is $\displaystyle\int_a^b \|\mathbf{x}'(t)\|\, dt$.

Arclength parameter: $\displaystyle s(t) = \int_a^t \|\mathbf{x}'(\tau)\|\, d\tau$.

Scalar line integral

Scalar line integral is an integral of a scalar function f over a path $\mathbf{x} : [a, b] \to \mathbb{R}^n$ of finite length relative to the arclength. It is defined as a limit of Riemann sums

$$\mathcal{S}(f, P, \tau_j) = \sum_{j=1}^{k} f(\mathbf{x}(\tau_j)) \left(s(t_j) - s(t_{j-1})\right),$$

where $P = \{t_0, t_1, \ldots, t_k\}$ is a partition of $[a, b]$, $\tau_j \in [t_j, t_{j-1}]$ for $1 \leq j \leq k$, and s is the arclength parameter of the path $\mathbf{x}$.

Theorem Let $\mathbf{x} : [a, b] \to \mathbb{R}^n$ be a smooth path and f be a function defined on the image of this path. Then

$$\int_{\mathbf{x}} f\, ds = \int_a^b f(\mathbf{x}(t))\, \|\mathbf{x}'(t)\|\, dt.$$

ds is referred to as the arclength element.

Vector line integral

Vector line integral is an integral of a vector field over a smooth path. It is a scalar.

Definition. Let $\mathbf{x} : [a, b] \to \mathbb{R}^n$ be a smooth path and $\mathbf{F}$ be a vector field defined on the image of this path. Then $\displaystyle\int_{\mathbf{x}} \mathbf{F} \cdot d\mathbf{s} = \int_a^b \mathbf{F}(\mathbf{x}(t)) \cdot \mathbf{x}'(t)\, dt.$

Alternatively, the integral of $\mathbf{F}$ over $\mathbf{x}$ can be represented as the integral of a **differential form**

$$\int_{\mathbf{x}} F_1\, dx_1 + F_2\, dx_2 + \cdots + F_n\, dx_n,$$

where $\mathbf{F} = (F_1, F_2, \ldots, F_n)$ and $dx_i = x_i'(t)\, dt$.

Line integrals and reparametrization

Given a path $\mathbf{x} : [a, b] \to \mathbb{R}^n$, we say that another path $\mathbf{y} : [c, d] \to \mathbb{R}^n$ is a **reparametrization** of $\mathbf{x}$ if there exists a continuous invertible function $u : [c, d] \to [a, b]$ such that $\mathbf{y}(t) = \mathbf{x}(u(t))$ for all $t \in [c, d]$.

The reparametrization may be orientation-preserving (when u is increasing) or orientation-reversing (when u is decreasing).

Theorem 1 Any scalar line integral is invariant under reparametrizations.

Theorem 2 Any vector line integral is invariant under orientation-preserving reparametrizations and changes its sign under orientation-reversing reparametrizations.

As a consequence, we can define the integral of a function over a simple curve and the integral of a vector field over a simple oriented curve.