MATH 311

Topics in Applied Mathematics I

Lecture 34:
Green's theorem.
Conservative vector fields.

Scalar line integral

Scalar line integral is an integral of a scalar function f over a path $\mathbf{x} : [a, b] \to \mathbb{R}^n$ of finite length relative to the arclength. It is defined as a limit of Riemann sums

$$\mathcal{S}(f, P, \tau_j) = \sum_{j=1}^{k} f(\mathbf{x}(\tau_j)) \left(s(t_j) - s(t_{j-1})\right),$$

where $P = \{t_0, t_1, \ldots, t_k\}$ is a partition of $[a, b]$, $\tau_j \in [t_j, t_{j-1}]$ for $1 \leq j \leq k$, and s is the arclength parameter of the path $\mathbf{x}$.

Theorem Let $\mathbf{x} : [a, b] \to \mathbb{R}^n$ be a smooth path and f be a function defined on the image of this path. Then

$$\int_{\mathbf{x}} f\, ds = \int_a^b f(\mathbf{x}(t))\, \|\mathbf{x}'(t)\|\, dt.$$

ds is referred to as the arclength element.

Vector line integral

Vector line integral is an integral of a vector field over a smooth path. It is a scalar.

Definition. Let $\mathbf{x} : [a, b] \to \mathbb{R}^n$ be a smooth path and $\mathbf{F}$ be a vector field defined on the image of this path. Then $\displaystyle\int_{\mathbf{x}} \mathbf{F} \cdot d\mathbf{s} = \int_a^b \mathbf{F}(\mathbf{x}(t)) \cdot \mathbf{x}'(t)\, dt$.

Alternatively, the integral of $\mathbf{F}$ over $\mathbf{x}$ can be represented as the integral of a **differential form**

$$\int_{\mathbf{x}} F_1\, dx_1 + F_2\, dx_2 + \cdots + F_n\, dx_n,$$

where $\mathbf{F} = (F_1, F_2, \dots, F_n)$ and $dx_i = x_i'(t)\, dt$.

Line integrals and reparametrization

Given a path $\mathbf{x} : [a, b] \to \mathbb{R}^n$, we say that another path $\mathbf{y} : [c, d] \to \mathbb{R}^n$ is a **reparametrization** of $\mathbf{x}$ if there exists a continuous invertible function $u : [c, d] \to [a, b]$ such that $\mathbf{y}(t) = \mathbf{x}(u(t))$ for all $t \in [c, d]$.

The reparametrization may be orientation-preserving (when u is increasing) or orientation-reversing (when u is decreasing).

Theorem 1 Any scalar line integral is invariant under reparametrizations.

Theorem 2 Any vector line integral is invariant under orientation-preserving reparametrizations and changes its sign under orientation-reversing reparametrizations.

As a consequence, we can define the integral of a function over a simple curve and the integral of a vector field over a simple oriented curve.

Applications of line integrals

- Mass of a wire

If f is the density on a wire C, then $\int_C f\, ds$ is the mass of C.

- Work of a force

If $\mathbf{F}$ is a force field, then $\int_{\mathbf{x}} \mathbf{F} \cdot d\mathbf{s}$ is the work done by $\mathbf{F}$ on a particle that moves along the path $\mathbf{x}$.

- Circulation of fluid

If $\mathbf{F}$ is the velocity field of a planar fluid, then the circulation of the fluid across a closed curve C is $\oint_C \mathbf{F} \cdot d\mathbf{s}$.

- Flux of fluid

If $\mathbf{F}$ is the velocity field of a planar fluid, then the flux of the fluid across a closed curve C is $\oint_C \mathbf{F} \cdot \mathbf{n}\, ds$, where $\mathbf{n}$ is the outward unit normal vector to C.

Green's Theorem

Theorem Let $D \subset \mathbb{R}^2$ be a closed, bounded region with piecewise smooth boundary ∂D oriented so that D is on the left as one traverses ∂D. Then for any smooth vector field $\mathbf{F} = (M, N)$ on D,

$$\oint_{\partial D} \mathbf{F} \cdot d\mathbf{s} = \iint_D \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) dx\, dy$$

or, equivalently,

$$\oint_{\partial D} M\, dx + N\, dy = \iint_D \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) dx\, dy.$$

Green's Theorem

Proof in the case $D = [0,1] \times [0,1]$ *and* $\mathbf{F} = (0, N)$*:*

$$\int_0^1 \frac{\partial N}{\partial x}(\xi, y)\, d\xi = N(1,y) - N(0,y)$$

for any $y \in [0,1]$ due to the Fundamental Theorem of Calculus. Integrating this equality by y over $[0,1]$, we obtain

$$\iint_D \frac{\partial N}{\partial x}\, dx\, dy = \int_0^1 N(1,y)\, dy - \int_0^1 N(0,y)\, dy.$$

Let $P_1 = (0,0)$, $P_2 = (1,0)$, $P_3 = (1,1)$, and $P_4 = (0,1)$. The first integral in the right-hand side equals the vector integral of the field $\mathbf{F}$ over the segment P_2P_3. The second integral equals the integral of $\mathbf{F}$ over the segment P_1P_4. Also, the integral of $\mathbf{F}$ over any horizontal segment is 0. It follows that the entire right-hand side equals the integral of $\mathbf{F}$ over the broken line $P_1P_2P_3P_4P_1$, that is, over ∂D.

Example

Consider vector fields $\mathbf{F}(x, y) = (-y, 0)$, $\mathbf{G}(x, y) = (0, x)$, and $\mathbf{H}(x, y) = (y, x)$.

According to Green's Theorem,

$$\oint_{\partial D} -y\,dx = \iint_D 1\,dx\,dy = \mathrm{area}(D),$$

$$\oint_{\partial D} x\,dy = \iint_D 1\,dx\,dy = \mathrm{area}(D),$$

$$\oint_{\partial D} y\,dx + x\,dy = \iint_D 0\,dx\,dy = 0.$$

Divergence Theorem

Theorem Let $D \subset \mathbb{R}^2$ be a closed, bounded region with piecewise smooth boundary ∂D oriented so that D is on the left as one traverses ∂D. Then for any smooth vector field $\mathbf{F}$ on D,

$$\oint_{\partial D} \mathbf{F} \cdot \mathbf{n} \, ds = \iint_D \nabla \cdot \mathbf{F} \, dA.$$

Proof: Let $\mathcal{L}$ denote the rotation of the plane $\mathbb{R}^2$ by 90° about the origin (counterclockwise). $\mathcal{L}$ is a linear transformation preserving the dot product. Therefore

$$\oint_{\partial D} \mathbf{F} \cdot \mathbf{n} \, ds = \oint_{\partial D} \mathcal{L}(\mathbf{F}) \cdot \mathcal{L}(\mathbf{n}) \, ds.$$

Note that $\mathcal{L}(\mathbf{n})$ is the unit tangent vector to ∂D. It follows that the right-hand side is the vector integral of $\mathcal{L}(\mathbf{F})$ over ∂D. If $\mathbf{F} = (M, N)$ then $\mathcal{L}(\mathbf{F}) = (-N, M)$. By Green's Theorem,

$$\oint_{\partial D} \mathcal{L}(\mathbf{F}) \cdot d\mathbf{s} = \oint_{\partial D} -N \, dx + M \, dy = \iint_D \left(\frac{\partial M}{\partial x} + \frac{\partial N}{\partial y} \right) dx \, dy.$$

Conservative vector fields

Let R be an open region in $\mathbb{R}^n$ such that any two points in R can be connected by a continuous path. Such regions are called **(arcwise) connected**.

Definition. A continuous vector field $\mathbf{F} : R \to \mathbb{R}^n$ is called **conservative** if $\displaystyle\int_{C_1} \mathbf{F} \cdot d\mathbf{s} = \int_{C_2} \mathbf{F} \cdot d\mathbf{s}$ for any two simple, piecewise smooth, oriented curves $C_1, C_2 \subset R$ with the same initial and terminal points.

An equivalent condition is that $\displaystyle\oint_C \mathbf{F} \cdot d\mathbf{s} = 0$ for any piecewise smooth closed curve $C \subset R$.

Conservative vector fields

Theorem The vector field $\mathbf{F}$ is conservative if and only if it is a gradient field, that is, $\mathbf{F} = \nabla f$ for some function $f : R \to \mathbb{R}$. If this is the case, then

$$\int_C \mathbf{F} \cdot d\mathbf{s} = f(B) - f(A)$$

for any piecewise smooth, oriented curve $C \subset R$ that connects the point A to the point B.

Remark. In the case $\mathbf{F}$ is a force field, conservativity means that energy is conserved. Moreover, in this case the function f is the potential energy.

Test of conservativity

Theorem If a smooth field $\mathbf{F} = (F_1, F_2, \dots, F_n)$ is conservative in a region $R \subset \mathbb{R}^n$, then the Jacobian matrix $\dfrac{\partial(F_1, F_2, \dots, F_n)}{\partial(x_1, x_2, \dots, x_n)}$ is symmetric everywhere in R, that is,

$$\frac{\partial F_i}{\partial x_j} = \frac{\partial F_j}{\partial x_i} \quad \text{for } i \neq j.$$

Indeed, if the field $\mathbf{F}$ is conservative, then $\mathbf{F} = \nabla f$ for some smooth function $f : R \to \mathbb{R}$. It follows that the Jacobian matrix of $\mathbf{F}$ is the **Hessian matrix** of f, that is, the matrix of second-order partial derivatives: $\dfrac{\partial F_i}{\partial x_j} = \dfrac{\partial^2 f}{\partial x_j \, \partial x_i}$.

Remark. The converse of the theorem holds provided that the region R is **simply-connected**, which means that any closed path in R can be continuously shrunk within R to a point.

Finding scalar potential

Example. $\mathbf{F}(x,y) = (2xy^3 + 3y\cos 3x,\ 3x^2y^2 + \sin 3x)$.

The vector field $\mathbf{F}$ is conservative if $\partial F_1/\partial y = \partial F_2/\partial x$.

$$\frac{\partial F_1}{\partial y} = 6xy^2 + 3\cos 3x, \quad \frac{\partial F_2}{\partial x} = 6xy^2 + 3\cos 3x.$$

Thus $\mathbf{F} = \nabla f$ for some function f (**scalar potential** of $\mathbf{F}$),

that is, $\dfrac{\partial f}{\partial x} = 2xy^3 + 3y\cos 3x, \quad \dfrac{\partial f}{\partial y} = 3x^2y^2 + \sin 3x.$

Integrating the second equality by y, we get

$$f(x,y) = \int (3x^2y^2 + \sin 3x)\,dy = x^2y^3 + y\sin 3x + g(x).$$

Substituting this into the first equality, we obtain that $2xy^3 + 3y\cos 3x + g'(x) = 2xy^3 + 3y\cos 3x$. Hence $g'(x) = 0$ so that $g(x) = c$, a constant. Then $f(x,y) = x^2y^3 + y\sin 3x + c$.