MATH 311

Topics in Applied Mathematics I

Lecture 36:

Gauss' theorem.

Stokes' theorem.

Surface integrals

Let $\mathbf{X}:D\to\mathbb{R}^3$ be a smooth parametrized surface, where $D\subset\mathbb{R}^2$ is a bounded region. Then for any continuous function $f:\mathbf{X}(D)\to\mathbb{R}$, the **scalar integral** of f over the surface \mathbf{X} is

$$\iint_{\mathbf{X}} f \, dS = \iint_{D} f(\mathbf{X}(s,t)) \, \left\| \frac{\partial \mathbf{X}}{\partial s} \times \frac{\partial \mathbf{X}}{\partial t} \right\| \, ds \, dt.$$

For any continuous vector field $\mathbf{F}: \mathbf{X}(D) \to \mathbb{R}^3$, the **vector integral** of \mathbf{F} along \mathbf{X} is

$$\iint_{\mathbf{X}} \mathbf{F} \cdot d\mathbf{S} = \iint_{D} \mathbf{F}(\mathbf{X}(s,t)) \cdot \left(\frac{\partial \mathbf{X}}{\partial s} \times \frac{\partial \mathbf{X}}{\partial t}\right) ds dt.$$

Equivalently,
$$\iint_{\mathbf{X}} \mathbf{F} \cdot d\mathbf{S} = \iint_{D} \begin{vmatrix} F_{1} & F_{2} & F_{3} \\ \frac{\partial X_{1}}{\partial s} & \frac{\partial X_{2}}{\partial s} & \frac{\partial X_{3}}{\partial s} \\ \frac{\partial X_{1}}{\partial t} & \frac{\partial X_{2}}{\partial t} & \frac{\partial X_{3}}{\partial t} \end{vmatrix} ds dt.$$

Surface integrals and reparametrization

Given two smooth parametrized surfaces $\mathbf{X}:D_1\to\mathbb{R}^3$ and $\mathbf{Y}:D_2\to\mathbb{R}^3$, we say that \mathbf{Y} is a **smooth reparametrization** of \mathbf{X} if there exists an invertible function $\mathbf{H}:D_2\to D_1$ such that $\mathbf{Y}=\mathbf{X}\circ\mathbf{H}$ and both \mathbf{H} and \mathbf{H}^{-1} are smooth.

Theorem 1 Any scalar surface integral is invariant under smooth reparametrizations.

Any smooth parametrization of a surface defines an **orientation** on it (continuous, unit normal vector field \mathbf{n}).

Theorem 2 Any vector surface integral is invariant under smooth orientation-preserving reparametrizations and changes its sign under orientation-reversing reparametrizations.

As a consequence, we can define the integral of a function over a non-parametrized smooth surface and the integral of a vector field along a non-parametrized, oriented smooth surface.

Gauss's Theorem

Corollary If a smooth vector field $\mathbf{F}: D \to \mathbb{R}^3$ has no divergence, $\nabla \cdot \mathbf{F} = 0$, then $\iint_C \mathbf{F} \cdot d\mathbf{S} = 0$ for any closed, piecewise smooth surface C that bounds a subregion of D.

Gauss' Theorem

Proof in the case $D = [0,1] \times [0,1] \times [0,1]$ and $\mathbf{F} = (0,0,P)$:

$$\int_0^1 \frac{\partial P}{\partial z}(x, y, \zeta) \, d\zeta = P(x, y, 1) - P(x, y, 0)$$

for any $x,y\in[0,1]$ due to the Fundamental Theorem of Calculus. Integrating this equality over the unit square $Q=[0,1]\times[0,1]$, we obtain

$$\iiint_{D} \frac{\partial P}{\partial z} dV = \iint_{Q} P(x, y, 1) dx dy - \iint_{Q} P(x, y, 0) dx dy.$$

The first integral in the right-hand side equals the integral of the field ${\bf F}$ along the top face $Q\times\{1\}$ of the cube D (oriented by the upward unit normals). The second integral equals the integral of ${\bf F}$ along the bottom face $Q\times\{0\}$ (oriented likewise). Note that integrals of ${\bf F}$ along the other faces of D are 0 (since ${\bf F}$ is parallel to those faces). It follows that the entire right-hand side equals the integral of ${\bf F}$ along ∂D .

Problem. Let C denote the closed cylinder with bottom given by z=0, top given by z=4, and lateral surface given by $x^2+y^2=9$. We orient ∂C with outward normals. Find the integral of a vector field $\mathbf{F}(x,y,z)=x\mathbf{e}_1+y\mathbf{e}_2+z\mathbf{e}_3$ along ∂C .

First let us evaluate the integral directly.

The top of the cylinder is parametrized by $\mathbf{X}_{\mathrm{top}}:D\to\mathbb{R}^3$, $\mathbf{X}_{\mathrm{top}}(x,y)=(x,y,4)$, where

$$D = \{(x,y) \in \mathbb{R}^2 : x^2 + y^2 \le 9\}.$$

The bottom is parametrized by $\mathbf{X}_{\mathrm{bot}}: D \to \mathbb{R}^3$, $\mathbf{X}_{\mathrm{bot}}(x,y) = (x,y,0)$.

The lateral surface is parametrized by $\mathbf{X}_{\mathrm{lat}}: [0, 2\pi] \times [0, 4] \to \mathbb{R}^3$, $\mathbf{X}_{\mathrm{lat}}(\phi, z) = (3\cos\phi, 3\sin\phi, z)$.

We have
$$\frac{\partial \mathbf{X}_{\mathrm{top}}}{\partial x} = (1,0,0)$$
, $\frac{\partial \mathbf{X}_{\mathrm{top}}}{\partial y} = (0,1,0)$. Hence $\frac{\partial \mathbf{X}_{\mathrm{top}}}{\partial x} \times \frac{\partial \mathbf{X}_{\mathrm{top}}}{\partial y} = \mathbf{e}_1 \times \mathbf{e}_2 = \mathbf{e}_3$.

Since
$$\mathbf{X}_{\mathrm{bot}} = \mathbf{X}_{\mathrm{top}} - (0, 0, 4)$$
, we also have $\frac{\partial \mathbf{X}_{\mathrm{bot}}}{\partial x} = \mathbf{e}_1$, $\frac{\partial \mathbf{X}_{\mathrm{bot}}}{\partial y} = \mathbf{e}_2$, and $\frac{\partial \mathbf{X}_{\mathrm{bot}}}{\partial x} \times \frac{\partial \mathbf{X}_{\mathrm{bot}}}{\partial y} = \mathbf{e}_3$.

Further, $\frac{\partial \mathbf{X}_{\text{lat}}}{\partial \phi} = (-3 \sin \phi, 3 \cos \phi, 0)$ and $\frac{\partial \mathbf{X}_{\text{lat}}}{\partial z} = (0, 0, 1)$. Therefore

$$\frac{\partial \mathbf{X}_{\mathrm{lat}}}{\partial \phi} \times \frac{\partial \mathbf{X}_{\mathrm{lat}}}{\partial z} = \begin{vmatrix} \mathbf{e}_1 & \mathbf{e}_2 & \mathbf{e}_3 \\ -3\sin\phi & 3\cos\phi & 0 \\ 0 & 0 & 1 \end{vmatrix} = (3\cos\phi, 3\sin\phi, 0).$$

We observe that X_{top} and X_{lat} agree with the orientation of the surface C while X_{bot} does not. It follows that

Integrating the vector field $\mathbf{F} = x\mathbf{e}_1 + y\mathbf{e}_2 + z\mathbf{e}_3$ along each part of the boundary of C, we obtain:

$$\iint_{\mathbf{X}_{\text{top}}} \mathbf{F} \cdot d\mathbf{S} = \iint_{D} (x, y, 4) \cdot (0, 0, 1) \, dx \, dy = \iint_{D} 4 \, dx \, dy = 36\pi,$$

$$\iint_{\mathbf{X}} \mathbf{F} \cdot d\mathbf{S} = \iint_{D} (x, y, 0) \cdot (0, 0, 1) \, dx \, dy = \iint_{D} 0 \, dx \, dy = 0,$$

$$\begin{split} &\iint_{\mathbf{X}_{\mathrm{lat}}} \mathbf{F} \cdot d\mathbf{S} = \\ &= \iint_{[0,2\pi] \times [0,4]} (3\cos\phi, 3\sin\phi, z) \cdot (3\cos\phi, 3\sin\phi, 0) \, d\phi \, dz \\ &= \iint_{[0,2\pi] \times [0,4]} 9 \, d\phi \, dz = 72\pi. \end{split}$$

Thus
$$\iint_{\partial C} \mathbf{F} \cdot d\mathbf{S} = 36\pi - 0 + 72\pi = 108\pi$$
.

Problem. Let C denote the closed cylinder with bottom given by z = 0, top given by z = 4, and lateral surface given by $x^2 + y^2 = 9$. We orient ∂C with outward normals. Find the integral of a vector field $\mathbf{F}(x, y, z) = x\mathbf{e}_1 + y\mathbf{e}_2 + z\mathbf{e}_3$ along ∂C .

Now let us use Gauss' Theorem:

$$\iint_{\partial C} \mathbf{F} \cdot d\mathbf{S} = \iiint_{C} \nabla \cdot \mathbf{F} \, dV$$

$$= \iiint_{C} \left(\frac{\partial}{\partial x} (x) + \frac{\partial}{\partial y} (y) + \frac{\partial}{\partial z} (z) \right) dx \, dy \, dz$$

$$= \iiint_{C} 3 \, dx \, dy \, dz = 108\pi.$$

Stokes's Theorem

Suppose S is an oriented surface in \mathbb{R}^3 bounded by an oriented curve ∂S . We say that ∂S is **oriented consistently with** S if, as one traverses ∂S , the surface S is on the left when looking down from the tip of \mathbf{n} , the unit normal vector indicating the orientation of S.

Theorem Let $S \subset \mathbb{R}^3$ be a bounded, piecewise smooth oriented surface with piecewise smooth boundary ∂S oriented consistently with S. Then for any smooth vector field \mathbf{F} on S,

$$\iint_{\mathcal{S}} \operatorname{curl}(\mathbf{F}) \cdot d\mathbf{S} = \oint_{\partial \mathcal{S}} \mathbf{F} \cdot d\mathbf{s}.$$

Corollary If the surface S is closed (i.e., has no boundary), then for any smooth vector field \mathbf{F} on S,

$$\iint_{S} \operatorname{curl}(\mathbf{F}) \cdot d\mathbf{S} = 0.$$

Example

Suppose that a bounded, piecewise smooth surface $S \subset \mathbb{R}^3$ is contained in the xy-coordinate plane, that is, $S = D \times \{0\}$ for a domain $D \subset \mathbb{R}^2$. We orient S by the upward unit normal vector $\mathbf{n} = (0,0,1)$ and orient the boundary $\partial S = \partial D \times \{0\}$ consistently with S. Further, suppose that \mathbf{F} is a horizontal vector field, $\mathbf{F} = (M,N,0)$. By Stokes' Theorem,

$$\iint_{S} \operatorname{curl}(\mathbf{F}) \cdot d\mathbf{S} = \oint_{\partial S} \mathbf{F} \cdot d\mathbf{s}.$$

Recall that $\iint_S \operatorname{curl}(\mathbf{F}) \cdot d\mathbf{S} = \iint_S \operatorname{curl}(\mathbf{F}) \cdot \mathbf{n} \ dS$. We obtain

$$\operatorname{curl}(\mathbf{F}) \cdot \mathbf{n} = \begin{vmatrix} 0 & 0 & 1 \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ M & N & 0 \end{vmatrix} = \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}.$$

It follows that this particular case of Stokes' Theorem is equivalent to Green's Theorem.