Sample problems for Test 2

Any problem may be altered or replaced by a different one!

Problem 1 Let $A = \begin{pmatrix} 0 & -1 & 4 & 1 \\ 1 & 1 & 2 & -1 \\ -3 & 0 & -1 & 0 \\ 2 & -1 & 0 & 1 \end{pmatrix}$.

(i) Find the rank and the nullity of the matrix A.
(ii) Find a basis for the row space of A, then extend this basis to a basis for $\mathbb{R}^4$.
(iii) Find a basis for the nullspace of A.

Problem 2 Let V be a subspace of $\mathcal{F}(\mathbb{R})$ spanned by functions e^x and e^{-x}. Let L be a linear operator on V such that

$$\begin{pmatrix} 2 & -1 \\ -3 & 2 \end{pmatrix}$$

is the matrix of L relative to the basis e^x, e^{-x}. Find the matrix of L relative to the basis $\cosh x = \frac{1}{2}(e^x + e^{-x})$, $\sinh x = \frac{1}{2}(e^x - e^{-x})$.

Problem 3 Let $L : V \to W$ be a linear mapping of a finite-dimensional vector space V to a vector space W. Show that

$$\dim \mathrm{Range}(L) + \dim \ker(L) = \dim V.$$

Problem 4 Let $A = \begin{pmatrix} 1 & 2 & 0 \\ 1 & 1 & 1 \\ 0 & 2 & 1 \end{pmatrix}$.

(i) Find all eigenvalues of the matrix A.
(ii) For each eigenvalue of A, find an associated eigenvector.
(iii) Is the matrix A diagonalizable? Explain.
(iv) Find all eigenvalues of the matrix A^2.

Problem 5 Find a linear polynomial which is the best least squares fit to the following data:

x	-2	-1	0	1	2
$f(x)$	-3	-2	1	2	5

Problem 6 Let V be a subspace of $\mathbb{R}^4$ spanned by the vectors $\mathbf{x}_1 = (1,1,1,1)$ and $\mathbf{x}_2 = (1,0,3,0)$.

(i) Find an orthonormal basis for V.
(ii) Find an orthonormal basis for the orthogonal complement $V^\perp$.