

MATH 311

Topics in Applied Mathematics I

**Lecture 7:**

**Inverse matrix (continued).**

**Transpose of a matrix.**

## Inverse matrix

*Definition.* Let  $A$  be an  $n \times n$  matrix. The **inverse** of  $A$  is an  $n \times n$  matrix, denoted  $A^{-1}$ , such that

$$\boxed{AA^{-1} = A^{-1}A = I.}$$

If  $A^{-1}$  exists then the matrix  $A$  is called **invertible**. Otherwise  $A$  is called **singular**.

System of  $n$  linear equations in  $n$  variables:

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2 \\ \dots\dots\dots \\ a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n = b_n \end{cases} \iff \mathbf{Ax} = \mathbf{b},$$

where

$$A = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{pmatrix}, \quad \mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}, \quad \mathbf{b} = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix}.$$

**Theorem** If the matrix  $A$  is invertible then the system has a unique solution, which is  $\mathbf{x} = A^{-1}\mathbf{b}$ .

## General results on inverse matrices

**Theorem 1** Given an  $n \times n$  matrix  $A$ , the following conditions are equivalent:

- (i)  $A$  is invertible;
- (ii)  $\mathbf{x} = \mathbf{0}$  is the only solution of the matrix equation  $A\mathbf{x} = \mathbf{0}$ ;
- (iii) the matrix equation  $A\mathbf{x} = \mathbf{b}$  has a unique solution for any  $n$ -dimensional column vector  $\mathbf{b}$ ;
- (iv) the row echelon form of  $A$  has no zero rows;
- (v) the reduced row echelon form of  $A$  is the identity matrix.

**Theorem 2** Suppose that a sequence of elementary row operations converts a matrix  $A$  into the identity matrix.

Then the same sequence of operations converts the identity matrix into the inverse matrix  $A^{-1}$ .

*Row echelon form of a square matrix:*

A 7x7 matrix in row echelon form. The diagonal elements are represented by squares, and the rest of the matrix is filled with asterisks. A blue staircase line starts at the top-left square and goes down and right to the bottom-right square, indicating that each row has a leading entry.

invertible case

A 7x7 matrix in row echelon form, similar to the first one. However, the asterisks in the third row, second column and the sixth row, seventh column are circled in red. A blue staircase line is also present, but it is shorter than in the first case, ending at the sixth row, indicating that the seventh row is a zero row.

noninvertible case

For any matrix in row echelon form, the number of columns with leading entries equals the number of rows with leading entries. For a square matrix, also the number of columns *without* leading entries (i.e., the number of free variables in a related system of linear equations) equals the number of rows *without* leading entries (i.e., zero rows).

*Row echelon form of a square matrix:*

$$\begin{pmatrix} \square & * & * & * & * & * & * \\ & \square & * & * & * & * & * \\ & & \square & * & * & * & * \\ & & & \square & * & * & * \\ & & & & \square & * & * \\ & & & & & \square & * \\ & & & & & & \square \end{pmatrix}$$

invertible case

$$\begin{pmatrix} \square & * & * & * & * & * & * \\ & \square & * & * & * & * & * \\ & & \square & * & * & * & * \\ & & & \square & * & * & * \\ & & & & \square & * & * \\ & & & & & \square & * \\ & & & & & & \square \end{pmatrix}$$

noninvertible case

Hence the row echelon form of a square matrix  $A$  is either strict triangular or else it has a zero row. In the former case, the equation  $A\mathbf{x} = \mathbf{b}$  *always* has a unique solution. In the latter case,  $A\mathbf{x} = \mathbf{b}$  *never* has a unique solution. Also, in the former case the reduced row echelon form of  $A$  is  $I$ .

*Example.*  $A = \begin{pmatrix} 3 & -2 & 0 \\ 1 & 0 & 1 \\ -2 & 3 & 0 \end{pmatrix}.$

*To check whether  $A$  is invertible, we convert it to row echelon form.*

Interchange the 1st row with the 2nd row:

$$\begin{pmatrix} 1 & 0 & 1 \\ 3 & -2 & 0 \\ -2 & 3 & 0 \end{pmatrix}$$

Add  $-3$  times the 1st row to the 2nd row:

$$\begin{pmatrix} 1 & 0 & 1 \\ 0 & -2 & -3 \\ -2 & 3 & 0 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 & 1 \\ 0 & -2 & -3 \\ -2 & 3 & 0 \end{pmatrix}$$

Add 2 times the 1st row to the 3rd row:

$$\begin{pmatrix} 1 & 0 & 1 \\ 0 & -2 & -3 \\ 0 & 3 & 2 \end{pmatrix}$$

Multiply the 2nd row by  $-0.5$ :

$$\begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1.5 \\ 0 & 3 & 2 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1.5 \\ 0 & 3 & 2 \end{pmatrix}$$

Add  $-3$  times the 2nd row to the 3rd row:

$$\begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1.5 \\ 0 & 0 & -2.5 \end{pmatrix}$$

Multiply the 3rd row by  $-0.4$ :

$$\begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1.5 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} \boxed{1} & 0 & 1 \\ 0 & \boxed{1} & 1.5 \\ 0 & 0 & \boxed{1} \end{pmatrix}$$

*We already know that the matrix  $A$  is invertible.*

*Let's proceed towards reduced row echelon form.*

Add  $-1.5$  times the 3rd row to the 2nd row:

$$\begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Add  $-1$  times the 3rd row to the 1st row:

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

To obtain  $A^{-1}$ , we need to apply the following sequence of elementary row operations to the identity matrix:

- interchange the 1st row with the 2nd row,
- add  $-3$  times the 1st row to the 2nd row,
- add 2 times the 1st row to the 3rd row,
- multiply the 2nd row by  $-0.5$ ,
- add  $-3$  times the 2nd row to the 3rd row,
- multiply the 3rd row by  $-0.4$ ,
- add  $-1.5$  times the 3rd row to the 2nd row,
- add  $-1$  times the 3rd row to the 1st row.

A convenient way to compute the inverse matrix  $A^{-1}$  is to merge the matrices  $A$  and  $I$  into one  $3 \times 6$  matrix  $(A | I)$ , and apply elementary row operations to this new matrix.

$$A = \begin{pmatrix} 3 & -2 & 0 \\ 1 & 0 & 1 \\ -2 & 3 & 0 \end{pmatrix}, \quad I = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$(A | I) = \left( \begin{array}{ccc|ccc} 3 & -2 & 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ -2 & 3 & 0 & 0 & 0 & 1 \end{array} \right)$$

$$\left( \begin{array}{ccc|ccc} 3 & -2 & 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ -2 & 3 & 0 & 0 & 0 & 1 \end{array} \right)$$

Interchange the 1st row with the 2nd row:

$$\left( \begin{array}{ccc|ccc} 1 & 0 & 1 & 0 & 1 & 0 \\ 3 & -2 & 0 & 1 & 0 & 0 \\ -2 & 3 & 0 & 0 & 0 & 1 \end{array} \right)$$

Add  $-3$  times the 1st row to the 2nd row:

$$\left( \begin{array}{ccc|ccc} 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & -2 & -3 & 1 & -3 & 0 \\ -2 & 3 & 0 & 0 & 0 & 1 \end{array} \right)$$

$$\left( \begin{array}{ccc|ccc} 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & -2 & -3 & 1 & -3 & 0 \\ -2 & 3 & 0 & 0 & 0 & 1 \end{array} \right)$$

Add 2 times the 1st row to the 3rd row:

$$\left( \begin{array}{ccc|ccc} 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & -2 & -3 & 1 & -3 & 0 \\ 0 & 3 & 2 & 0 & 2 & 1 \end{array} \right)$$

Multiply the 2nd row by  $-0.5$ :

$$\left( \begin{array}{ccc|ccc} 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1.5 & -0.5 & 1.5 & 0 \\ 0 & 3 & 2 & 0 & 2 & 1 \end{array} \right)$$

$$\left( \begin{array}{ccc|ccc} 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1.5 & -0.5 & 1.5 & 0 \\ 0 & 3 & 2 & 0 & 2 & 1 \end{array} \right)$$

Add  $-3$  times the 2nd row to the 3rd row:

$$\left( \begin{array}{ccc|ccc} 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1.5 & -0.5 & 1.5 & 0 \\ 0 & 0 & -2.5 & 1.5 & -2.5 & 1 \end{array} \right)$$

Multiply the 3rd row by  $-0.4$ :

$$\left( \begin{array}{ccc|ccc} 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1.5 & -0.5 & 1.5 & 0 \\ 0 & 0 & 1 & -0.6 & 1 & -0.4 \end{array} \right)$$

$$\left( \begin{array}{ccc|ccc} 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1.5 & -0.5 & 1.5 & 0 \\ 0 & 0 & 1 & -0.6 & 1 & -0.4 \end{array} \right)$$

Add  $-1.5$  times the 3rd row to the 2nd row:

$$\left( \begin{array}{ccc|ccc} 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0.4 & 0 & 0.6 \\ 0 & 0 & 1 & -0.6 & 1 & -0.4 \end{array} \right)$$

Add  $-1$  times the 3rd row to the 1st row:

$$\left( \begin{array}{ccc|ccc} 1 & 0 & 0 & 0.6 & 0 & 0.4 \\ 0 & 1 & 0 & 0.4 & 0 & 0.6 \\ 0 & 0 & 1 & -0.6 & 1 & -0.4 \end{array} \right) = (I | A^{-1})$$

Thus 
$$\begin{pmatrix} 3 & -2 & 0 \\ 1 & 0 & 1 \\ -2 & 3 & 0 \end{pmatrix}^{-1} = \begin{pmatrix} \frac{3}{5} & 0 & \frac{2}{5} \\ \frac{2}{5} & 0 & \frac{3}{5} \\ -\frac{3}{5} & 1 & -\frac{2}{5} \end{pmatrix}.$$

That is,

$$\begin{pmatrix} 3 & -2 & 0 \\ 1 & 0 & 1 \\ -2 & 3 & 0 \end{pmatrix} \begin{pmatrix} \frac{3}{5} & 0 & \frac{2}{5} \\ \frac{2}{5} & 0 & \frac{3}{5} \\ -\frac{3}{5} & 1 & -\frac{2}{5} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

$$\begin{pmatrix} \frac{3}{5} & 0 & \frac{2}{5} \\ \frac{2}{5} & 0 & \frac{3}{5} \\ -\frac{3}{5} & 1 & -\frac{2}{5} \end{pmatrix} \begin{pmatrix} 3 & -2 & 0 \\ 1 & 0 & 1 \\ -2 & 3 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

## Why does it work?

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{pmatrix} = \begin{pmatrix} a_1 & a_2 & a_3 \\ 2b_1 & 2b_2 & 2b_3 \\ c_1 & c_2 & c_3 \end{pmatrix},$$

$$\begin{pmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{pmatrix} = \begin{pmatrix} a_1 & a_2 & a_3 \\ b_1+3a_1 & b_2+3a_2 & b_3+3a_3 \\ c_1 & c_2 & c_3 \end{pmatrix},$$

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{pmatrix} = \begin{pmatrix} a_1 & a_2 & a_3 \\ c_1 & c_2 & c_3 \\ b_1 & b_2 & b_3 \end{pmatrix}.$$

**Proposition** Any elementary row operation can be simulated as left multiplication by a certain matrix.



## Elementary matrices

$$E = \begin{pmatrix} 1 & & & & & & & \\ \vdots & \ddots & & & & & & \\ 0 & \cdots & 1 & & & & & \\ \vdots & & \vdots & \ddots & & & & \\ 0 & \cdots & r & \cdots & 1 & & & \\ \vdots & & \vdots & & \vdots & \ddots & & \\ 0 & \cdots & 0 & \cdots & 0 & \cdots & 1 & \end{pmatrix} \begin{array}{l} \text{row \#}i \\ \\ \text{row \#}j \end{array}$$

To obtain the matrix  $EA$  from  $A$ , add  $r$  times the  $i$ th row to the  $j$ th row. To obtain the matrix  $AE$  from  $A$ , add  $r$  times the  $j$ th column to the  $i$ th column.



## Why does it work? (continued)

Assume that a square matrix  $A$  can be converted to the identity matrix by a sequence of elementary row operations. Then  $E_k E_{k-1} \dots E_2 E_1 A = I$ , where  $E_1, E_2, \dots, E_k$  are elementary matrices simulating those operations.

Applying the same sequence of operations to the identity matrix, we obtain the matrix

$$B = E_k E_{k-1} \dots E_2 E_1 I = E_k E_{k-1} \dots E_2 E_1.$$

Thus  $BA = I$ . Besides,  $B$  is invertible since elementary matrices are invertible (why?). It follows that  $A = B^{-1}$ , then  $B = A^{-1}$ .

## Transpose of a matrix

*Definition.* Given a matrix  $A$ , the **transpose** of  $A$ , denoted  $A^T$ , is the matrix whose rows are columns of  $A$  (and whose columns are rows of  $A$ ). That is, if  $A = (a_{ij})$  then  $A^T = (b_{ij})$ , where  $b_{ij} = a_{ji}$ .

*Examples.* 
$$\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix}^T = \begin{pmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{pmatrix},$$

$$\begin{pmatrix} 7 \\ 8 \\ 9 \end{pmatrix}^T = (7, 8, 9), \quad \begin{pmatrix} 4 & 7 \\ 7 & 0 \end{pmatrix}^T = \begin{pmatrix} 4 & 7 \\ 7 & 0 \end{pmatrix}.$$

## Properties of transposes:

- $(A^T)^T = A$
- $(A + B)^T = A^T + B^T$
- $(rA)^T = rA^T$
- $(AB)^T = B^T A^T$
- $(A_1 A_2 \dots A_k)^T = A_k^T \dots A_2^T A_1^T$
- $(A^{-1})^T = (A^T)^{-1}$

*Definition.* A square matrix  $A$  is said to be **symmetric** if  $A^T = A$ .

For example, any diagonal matrix is symmetric.

**Proposition** For any square matrix  $A$  the matrices  $B = AA^T$  and  $C = A + A^T$  are symmetric.

*Proof:*

$$B^T = (AA^T)^T = (A^T)^T A^T = AA^T = B,$$

$$C^T = (A + A^T)^T = A^T + (A^T)^T = A^T + A = C.$$