MATH 311

Topics in Applied Mathematics I

**Lecture 9:
Evaluation of determinants.
Cramer's rule.**

Determinants: definition in low dimensions

Definition. $\det(a) = a$, $\quad \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$,

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32}$$
$$- a_{13}a_{22}a_{31} - a_{12}a_{21}a_{33} - a_{11}a_{23}a_{32}.$$

$$+ : \begin{pmatrix} \boxed{*} & * & * \\ * & \boxed{*} & * \\ * & * & \boxed{*} \end{pmatrix}, \begin{pmatrix} * & \boxed{*} & * \\ * & * & \boxed{*} \\ \boxed{*} & * & * \end{pmatrix}, \begin{pmatrix} * & * & \boxed{*} \\ \boxed{*} & * & * \\ * & \boxed{*} & * \end{pmatrix}.$$

$$- : \begin{pmatrix} * & * & \boxed{*} \\ * & \boxed{*} & * \\ \boxed{*} & * & * \end{pmatrix}, \begin{pmatrix} * & \boxed{*} & * \\ \boxed{*} & * & * \\ * & * & \boxed{*} \end{pmatrix}, \begin{pmatrix} \boxed{*} & * & * \\ * & * & \boxed{*} \\ * & \boxed{*} & * \end{pmatrix}.$$

Properties of determinants

Determinants and elementary row operations:

- if a row of a matrix is multiplied by a scalar r, the determinant is also multiplied by r;
- if we add a row of a matrix multiplied by a scalar to another row, the determinant remains the same;
- if we interchange two rows of a matrix, the determinant changes its sign.

Properties of determinants

Tests for singularity:

- if a matrix A has a zero row then $\det A = 0$;

- if a matrix A has two identical rows then $\det A = 0$;

- if a matrix A has two proportional rows then $\det A = 0$;

- if a matrix A is not invertible then $\det A = 0$.

Properties of determinants

Special matrices:

- $\det I = 1$;

- the determinant of a diagonal matrix is equal to the product of its diagonal entries;

- the determinant of an upper triangular matrix is equal to the product of its diagonal entries.

Properties of determinants

Determinant of the transpose:
- If A is a square matrix then $\det A^T = \det A$.

Columns vs. rows:
- if one column of a matrix is multiplied by a scalar, the determinant is multiplied by the same scalar;
- adding a scalar multiple of one column to another does not change the determinant;
- interchanging two columns of a matrix changes the sign of its determinant;
- if a matrix A has a zero column or two proportional columns then $\det A = 0$.

Row and column expansions

Given an $n \times n$ matrix $A = (a_{ij})$, let M_{ij} denote the $(n-1) \times (n-1)$ submatrix obtained by deleting the ith row and the jth column of A.

Theorem For any $1 \leq k, m \leq n$ we have that

$$\det A = \sum_{j=1}^{n} (-1)^{k+j} a_{kj} \det M_{kj},$$

(*expansion by kth row*)

$$\det A = \sum_{i=1}^{n} (-1)^{i+m} a_{im} \det M_{im}.$$

(*expansion by mth column*)

Signs for row/column expansions

$$\begin{pmatrix} + & - & + & - & \cdots \\ - & + & - & + & \cdots \\ + & - & + & - & \cdots \\ - & + & - & + & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

Evaluation of determinants

Example. $B = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 13 \end{pmatrix}$.

First let's do some row reduction.

Add -4 times the 1st row to the 2nd row:

$$\begin{vmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 13 \end{vmatrix} = \begin{vmatrix} 1 & 2 & 3 \\ 0 & -3 & -6 \\ 7 & 8 & 13 \end{vmatrix}$$

Add -7 times the 1st row to the 3rd row:

$$\begin{vmatrix} 1 & 2 & 3 \\ 0 & -3 & -6 \\ 7 & 8 & 13 \end{vmatrix} = \begin{vmatrix} 1 & 2 & 3 \\ 0 & -3 & -6 \\ 0 & -6 & -8 \end{vmatrix}$$

Expand the determinant by the 1st column:

$$\begin{vmatrix} 1 & 2 & 3 \\ 0 & -3 & -6 \\ 0 & -6 & -8 \end{vmatrix} = 1 \begin{vmatrix} -3 & -6 \\ -6 & -8 \end{vmatrix}$$

Thus

$$\det B = \begin{vmatrix} -3 & -6 \\ -6 & -8 \end{vmatrix} = (-3) \begin{vmatrix} 1 & 2 \\ -6 & -8 \end{vmatrix}$$

$$= (-3)(-2) \begin{vmatrix} 1 & 2 \\ 3 & 4 \end{vmatrix} = (-3)(-2)(-2) = -12.$$

Example. $C = \begin{pmatrix} 2 & -2 & 0 & 3 \\ -5 & 3 & 2 & 1 \\ 1 & -1 & 0 & -3 \\ 2 & 0 & 0 & -1 \end{pmatrix}$, $\det C = ?$

Expand the determinant by the 3rd column:

$$\begin{vmatrix} 2 & -2 & 0 & 3 \\ -5 & 3 & 2 & 1 \\ 1 & -1 & 0 & -3 \\ 2 & 0 & 0 & -1 \end{vmatrix} = -2 \begin{vmatrix} 2 & -2 & 3 \\ 1 & -1 & -3 \\ 2 & 0 & -1 \end{vmatrix}$$

Add -2 times the 2nd row to the 1st row:

$$\det C = -2 \begin{vmatrix} 2 & -2 & 3 \\ 1 & -1 & -3 \\ 2 & 0 & -1 \end{vmatrix} = -2 \begin{vmatrix} 0 & 0 & 9 \\ 1 & -1 & -3 \\ 2 & 0 & -1 \end{vmatrix}$$

Expand the determinant by the 1st row:

$$\det C = -2 \begin{vmatrix} 0 & 0 & 9 \\ 1 & -1 & -3 \\ 2 & 0 & -1 \end{vmatrix} = -2 \cdot 9 \begin{vmatrix} 1 & -1 \\ 2 & 0 \end{vmatrix}$$

Thus

$$\det C = -18 \begin{vmatrix} 1 & -1 \\ 2 & 0 \end{vmatrix} = -18 \cdot 2 = -36.$$

Problem. For what values of *a* will the following system have a unique solution?

$$\begin{cases} x + 2y + z = 1 \\ -x + 4y + 2z = 2 \\ 2x - 2y + az = 3 \end{cases}$$

The system has a unique solution if and only if the coefficient matrix is invertible.

$$A = \begin{pmatrix} 1 & 2 & 1 \\ -1 & 4 & 2 \\ 2 & -2 & a \end{pmatrix}, \quad \det A = ?$$

$$A = \begin{pmatrix} 1 & 2 & 1 \\ -1 & 4 & 2 \\ 2 & -2 & a \end{pmatrix}, \quad \det A = ?$$

Add -2 times the 3rd column to the 2nd column:

$$\begin{vmatrix} 1 & 2 & 1 \\ -1 & 4 & 2 \\ 2 & -2 & a \end{vmatrix} = \begin{vmatrix} 1 & 0 & 1 \\ -1 & 0 & 2 \\ 2 & -2-2a & a \end{vmatrix}$$

Expand the determinant by the 2nd column:

$$\det A = \begin{vmatrix} 1 & 0 & 1 \\ -1 & 0 & 2 \\ 2 & -2-2a & a \end{vmatrix} = -(-2-2a)\begin{vmatrix} 1 & 1 \\ -1 & 2 \end{vmatrix}$$

Hence $\det A = -(-2-2a) \cdot 3 = 6(1+a)$.

Thus A is invertible if and only if $a \neq -1$.

More properties of determinants

Determinants and matrix multiplication:

- if A and B are $n \times n$ matrices then
$$\det(AB) = \det A \cdot \det B;$$

- if A and B are $n \times n$ matrices then
$$\det(AB) = \det(BA);$$

- if A is an invertible matrix then
$$\det(A^{-1}) = (\det A)^{-1}.$$

Determinants and scalar multiplication:

- if A is an $n \times n$ matrix and $r \in \mathbb{R}$ then
$$\det(rA) = r^n \det A.$$

Examples

$$X = \begin{pmatrix} -1 & 2 & 1 \\ 0 & 2 & -2 \\ 0 & 0 & -3 \end{pmatrix}, \ \ Y = \begin{pmatrix} 1 & 0 & 0 \\ -1 & 3 & 0 \\ 2 & -2 & 1 \end{pmatrix}.$$

$\det X = (-1) \cdot 2 \cdot (-3) = 6, \quad \det Y = \det Y^T = 3,$

$\det(XY) = 6 \cdot 3 = 18, \quad \det(YX) = 3 \cdot 6 = 18,$

$\det(Y^{-1}) = 1/3, \quad \det(XY^{-1}) = 6/3 = 2,$

$\det(XYX^{-1}) = \det Y = 3, \quad \det(X^{-1}Y^{-1}XY) = 1,$

$\det(2X) = 2^3 \det X = 2^3 \cdot 6 = 48,$

$\det(-3X^T XY^{-4}) = (-3)^3 \cdot 6 \cdot 6 \cdot 3^{-4} = -12.$

Let us try to find a solution of a general system of 2 linear equations in 2 variables:

$$\begin{cases} a_{11}x + a_{12}y = b_1, \\ a_{21}x + a_{22}y = b_2. \end{cases}$$

Solve the 1st equation for x: $x = (b_1 - a_{12}y)/a_{11}$.

Substitute into the 2nd equation:

$$a_{21}(b_1 - a_{12}y)/a_{11} + a_{22}y = b_2.$$

Solve for y: $y = \dfrac{a_{11}b_2 - a_{21}b_1}{a_{11}a_{22} - a_{12}a_{21}}$.

Back substitution: $x = (b_1 - a_{12}y)/a_{11} = \dfrac{a_{22}b_1 - a_{12}b_2}{a_{11}a_{22} - a_{12}a_{21}}$.

Thus

$$x = \frac{\begin{vmatrix} b_1 & a_{12} \\ b_2 & a_{22} \end{vmatrix}}{\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}}, \qquad y = \frac{\begin{vmatrix} a_{11} & b_1 \\ a_{21} & b_2 \end{vmatrix}}{\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}}.$$

Cramer's rule

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2 \\ \qquad \cdots\cdots\cdots \\ a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n = b_n \end{cases} \iff A\mathbf{x} = \mathbf{b}$$

Theorem Assume that the matrix A is invertible. Then the only solution of the system is given by

$$x_i = \frac{\det A_i}{\det A}, \quad i = 1, 2, \ldots, n,$$

where the matrix A_i is obtained by substituting the vector $\mathbf{b}$ for the ith column of A.

Determinants and the inverse matrix

Given an $n \times n$ matrix $A = (a_{ij})$, let M_{ij} denote the $(n-1) \times (n-1)$ submatrix obtained by deleting the ith row and the jth column of A. The **cofactor matrix** of A is an $n \times n$ matrix $\widetilde{A} = (\alpha_{ij})$ defined by $\alpha_{ij} = (-1)^{i+j} \det M_{ij}$.

Theorem $\widetilde{A}^T A = A \widetilde{A}^T = (\det A)I$.

Sketch of the proof: $A \widetilde{A}^T = (\det A)I$ means that

$$\sum_{j=1}^{n} (-1)^{k+j} a_{kj} \det M_{kj} = \det A \quad \text{for all } k,$$

$$\sum_{j=1}^{n} (-1)^{k+j} a_{mj} \det M_{kj} = 0 \quad \text{for } m \neq k.$$

Indeed, the 1st equality is the expansion of $\det A$ by the kth row. The 2nd equality is an analogous expansion of $\det B$, where the matrix B is obtained from A by replacing its kth row with a copy of the mth row (clearly, $\det B = 0$). $\widetilde{A}^T A = (\det A)I$ is verified similarly, using column expansions.

Corollary If $\det A \neq 0$ then $A^{-1} = (\det A)^{-1} \widetilde{A}^T$.