MATH 311

Topics in Applied Mathematics I

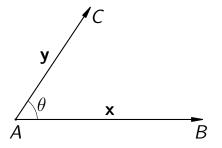
Lecture 24: Orthogonal complement. Orthogonal projection.

Beyond linearity: Euclidean structure

The vector space \mathbb{R}^n is also a **Euclidean space**.

The Euclidean structure includes:

- length of a vector: |x|,
- ullet angle between vectors: heta,
- dot product: $\mathbf{x} \cdot \mathbf{y} = |\mathbf{x}| |\mathbf{y}| \cos \theta$.



Length and distance

Definition. The **length** of a vector
$$\mathbf{v} = (v_1, v_2, \dots, v_n) \in \mathbb{R}^n$$
 is $\|\mathbf{v}\| = \sqrt{v_1^2 + v_2^2 + \dots + v_n^2}$.

The **distance** between vectors \mathbf{x} and \mathbf{y} is defined as $\|\mathbf{y} - \mathbf{x}\|$.

Properties of length:

$$\|\mathbf{x}\| \geq 0$$
, $\|\mathbf{x}\| = 0$ only if $\mathbf{x} = \mathbf{0}$ (positivity) $\|r\mathbf{x}\| = |r| \|\mathbf{x}\|$ (homogeneity) $\|\mathbf{x} + \mathbf{y}\| \leq \|\mathbf{x}\| + \|\mathbf{y}\|$ (triangle inequality)

Scalar product

Definition. The **scalar product** of vectors $\mathbf{x} = (x_1, x_2, \dots, x_n)$ and $\mathbf{y} = (y_1, y_2, \dots, y_n)$ is $\mathbf{x} \cdot \mathbf{y} = x_1 y_1 + x_2 y_2 + \dots + x_n y_n$.

Alternative notation: (\mathbf{x}, \mathbf{y}) or (\mathbf{x}, \mathbf{y}) .

Properties of scalar product:

$$\mathbf{x} \cdot \mathbf{x} \ge 0$$
, $\mathbf{x} \cdot \mathbf{x} = 0$ only if $\mathbf{x} = \mathbf{0}$ (positivity)
 $\mathbf{x} \cdot \mathbf{y} = \mathbf{y} \cdot \mathbf{x}$ (symmetry)
 $(\mathbf{x} + \mathbf{y}) \cdot \mathbf{z} = \mathbf{x} \cdot \mathbf{z} + \mathbf{y} \cdot \mathbf{z}$ (distributive law)
 $(r\mathbf{x}) \cdot \mathbf{y} = r(\mathbf{x} \cdot \mathbf{y})$ (homogeneity)

In particular, $\mathbf{x} \cdot \mathbf{y}$ is a **bilinear** function (i.e., it is both a linear function of \mathbf{x} and a linear function of \mathbf{y}).

Angle

Cauchy-Schwarz inequality:
$$|\mathbf{x} \cdot \mathbf{y}| \leq ||\mathbf{x}|| \, ||\mathbf{y}||$$
.

By the Cauchy-Schwarz inequality, for any nonzero vectors $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$ we have

$$\cos \theta = \frac{\mathbf{x} \cdot \mathbf{y}}{\|\mathbf{x}\| \|\mathbf{y}\|}$$
 for a unique $0 \le \theta \le \pi$.

 θ is called the **angle** between the vectors **x** and **y**. The vectors **x** and **y** are said to be **orthogonal** (denoted $\mathbf{x} \perp \mathbf{y}$) if $\mathbf{x} \cdot \mathbf{y} = 0$ (i.e., if $\theta = 90^{\circ}$).

Orthogonality

Definition 1. Vectors $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$ are said to be **orthogonal** (denoted $\mathbf{x} \perp \mathbf{y}$) if $\mathbf{x} \cdot \mathbf{y} = 0$.

Definition 2. A vector $\mathbf{x} \in \mathbb{R}^n$ is said to be **orthogonal** to a nonempty set $Y \subset \mathbb{R}^n$ (denoted $\mathbf{x} \perp Y$) if $\mathbf{x} \cdot \mathbf{y} = 0$ for any $\mathbf{y} \in Y$.

Definition 3. Nonempty sets $X, Y \subset \mathbb{R}^n$ are said to be **orthogonal** (denoted $X \perp Y$) if $\mathbf{x} \cdot \mathbf{y} = 0$ for any $\mathbf{x} \in X$ and $\mathbf{y} \in Y$.

Examples in \mathbb{R}^3 . • The line x = y = 0 is orthogonal to the line y = z = 0.

Indeed, if $\mathbf{v} = (0, 0, z)$ and $\mathbf{w} = (x, 0, 0)$ then $\mathbf{v} \cdot \mathbf{w} = 0$.

• The line x = y = 0 is orthogonal to the plane z = 0.

Indeed, if $\mathbf{v} = (0, 0, z)$ and $\mathbf{w} = (x, y, 0)$ then $\mathbf{v} \cdot \mathbf{w} = 0$.

• The line x = y = 0 is not orthogonal to the plane z = 1.

The vector $\mathbf{v} = (0,0,1)$ belongs to both the line and the plane, and $\mathbf{v} \cdot \mathbf{v} = 1 \neq 0$.

• The plane z = 0 is not orthogonal to the plane y = 0.

The vector $\mathbf{v} = (1, 0, 0)$ belongs to both planes and $\mathbf{v} \cdot \mathbf{v} = 1 \neq 0$.

Proposition 1 If $X, Y \in \mathbb{R}^n$ are orthogonal sets then either they are disjoint or $X \cap Y = \{0\}$.

$$\textit{Proof:} \quad \mathbf{v} \in X \cap Y \implies \mathbf{v} \perp \mathbf{v} \implies \mathbf{v} \cdot \mathbf{v} = 0 \implies \mathbf{v} = \mathbf{0}.$$

Proposition 2 Let V be a subspace of \mathbb{R}^n and S be a spanning set for V. Then for any $\mathbf{x} \in \mathbb{R}^n$

Proof: Any
$$\mathbf{v} \in V$$
 is represented as $\mathbf{v} = a_1 \mathbf{v}_1 + \cdots + a_k \mathbf{v}_k$,

 $x \perp S \implies x \perp V$.

where $\mathbf{v}_i \in S$ and $a_i \in \mathbb{R}$. If $\mathbf{x} \perp S$ then

$$\mathbf{x} \cdot \mathbf{v} = a_1(\mathbf{x} \cdot \mathbf{v}_1) + \cdots + a_k(\mathbf{x} \cdot \mathbf{v}_k) = 0 \implies \mathbf{x} \perp \mathbf{v}.$$

Example. The vector $\mathbf{v} = (1,1,1)$ is orthogonal to the plane spanned by vectors $\mathbf{w}_1 = (2,-3,1)$ and $\mathbf{w}_2 = (0,1,-1)$ (because $\mathbf{v} \cdot \mathbf{w}_1 = \mathbf{v} \cdot \mathbf{w}_2 = 0$).

Orthogonal complement

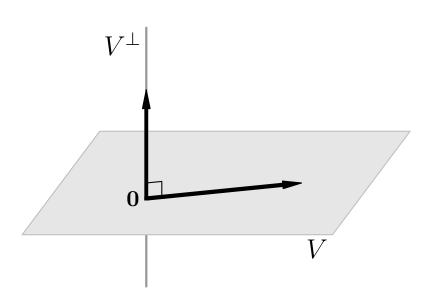
Definition. Let $S \subset \mathbb{R}^n$. The **orthogonal complement** of S, denoted S^{\perp} , is the set of all vectors $\mathbf{x} \in \mathbb{R}^n$ that are orthogonal to S. That is, S^{\perp} is the largest subset of \mathbb{R}^n orthogonal to S.

Theorem 1 S^{\perp} is a subspace of \mathbb{R}^n .

Note that $S \subset (S^{\perp})^{\perp}$, hence $\mathrm{Span}(S) \subset (S^{\perp})^{\perp}$.

Theorem 2 $(S^{\perp})^{\perp} = \operatorname{Span}(S)$. In particular, for any subspace V we have $(V^{\perp})^{\perp} = V$.

Example. Consider a line $L = \{(x,0,0) \mid x \in \mathbb{R}\}$ and a plane $\Pi = \{(0,y,z) \mid y,z \in \mathbb{R}\}$ in \mathbb{R}^3 . Then $L^{\perp} = \Pi$ and $\Pi^{\perp} = L$.



Fundamental subspaces

Definition. Given an $m \times n$ matrix A, let

$$N(A) = \{\mathbf{x} \in \mathbb{R}^n \mid A\mathbf{x} = \mathbf{0}\},$$

$$R(A) = \{ \mathbf{b} \in \mathbb{R}^m \mid \mathbf{b} = A\mathbf{x} \text{ for some } \mathbf{x} \in \mathbb{R}^n \}.$$

R(A) is the range of a linear mapping $L: \mathbb{R}^n \to \mathbb{R}^m$, $L(\mathbf{x}) = A\mathbf{x}$. N(A) is the kernel of L.

Also, N(A) is the nullspace of the matrix A while R(A) is the column space of A. The row space of A is $R(A^T)$.

The subspaces $N(A), R(A^T) \subset \mathbb{R}^n$ and $R(A), N(A^T) \subset \mathbb{R}^m$ are **fundamental subspaces** associated to the matrix A.

Theorem $N(A) = R(A^T)^{\perp}$, $N(A^T) = R(A)^{\perp}$. That is, the nullspace of a matrix is the orthogonal complement of its row space.

Proof: The equality $A\mathbf{x} = \mathbf{0}$ means that the vector \mathbf{x} is orthogonal to rows of the matrix A. Therefore $N(A) = S^{\perp}$, where S is the set of rows of A. It remains to note that $S^{\perp} = \operatorname{Span}(S)^{\perp} = R(A^{T})^{\perp}$.

Corollary Let V be a subspace of \mathbb{R}^n . Then dim $V + \dim V^{\perp} = n$.

Proof: Pick a basis $\mathbf{v}_1, \ldots, \mathbf{v}_k$ for V. Let A be the $k \times n$ matrix whose rows are vectors $\mathbf{v}_1, \ldots, \mathbf{v}_k$. Then $V = R(A^T)$, hence $V^{\perp} = N(A)$. Consequently, dim V and dim V^{\perp} are rank and nullity of A. Therefore dim $V + \dim V^{\perp}$ equals the number of columns of A, which is n.

Problem. Let V be the plane spanned by vectors $\mathbf{v}_1 = (1,1,0)$ and $\mathbf{v}_2 = (0,1,1)$. Find V^{\perp} .

The orthogonal complement to V is the same as the orthogonal complement of the set $\{\mathbf{v}_1,\mathbf{v}_2\}$. A vector $\mathbf{u}=(x,y,z)$ belongs to the latter if and only if

$$\begin{cases} \mathbf{u} \cdot \mathbf{v}_1 = 0 \\ \mathbf{u} \cdot \mathbf{v}_2 = 0 \end{cases} \iff \begin{cases} x + y = 0 \\ y + z = 0 \end{cases}$$

Alternatively, the subspace V is the row space of the matrix

$$A = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix},$$

hence V^{\perp} is the nullspace of A.

The general solution of the system (or, equivalently, the general element of the nullspace of A) is (t, -t, t) = t(1, -1, 1), $t \in \mathbb{R}$. Thus V^{\perp} is the straight line spanned by the vector (1, -1, 1).

Orthogonal projection

Theorem 1 Let V be a subspace of \mathbb{R}^n . Then any vector $\mathbf{x} \in \mathbb{R}^n$ is uniquely represented as $\mathbf{x} = \mathbf{p} + \mathbf{o}$, where $\mathbf{p} \in V$ and $\mathbf{o} \in V^{\perp}$.

Idea of the proof: Let $\mathbf{v}_1, \ldots, \mathbf{v}_k$ be a basis for V and $\mathbf{w}_1, \ldots, \mathbf{w}_m$ be a basis for V^{\perp} . Then $\mathbf{v}_1, \ldots, \mathbf{v}_k, \mathbf{w}_1, \ldots, \mathbf{w}_m$ is a linearly independent set. Hence it is a basis for \mathbb{R}^n .

In the above expansion, \mathbf{p} is called the **orthogonal projection** of the vector \mathbf{x} onto the subspace V.

Theorem 2 $\|\mathbf{x} - \mathbf{v}\| > \|\mathbf{x} - \mathbf{p}\|$ for any $\mathbf{v} \neq \mathbf{p}$ in V.

Thus $\|\mathbf{o}\| = \|\mathbf{x} - \mathbf{p}\| = \min_{\mathbf{v} \in V} \|\mathbf{x} - \mathbf{v}\|$ is the **distance** from the vector \mathbf{x} to the subspace V.

