MATH 311

Topics in Applied Mathematics I

Lecture 33:
Area and volume.
Multiple integrals.

Let $\mathcal{P}$ be the smallest collection of subsets of $\mathbb{R}^2$ such that it contains all polygons and if $X, Y \in \mathcal{P}$, then $X \cup Y, X \cap Y, X \setminus Y \in \mathcal{P}$.

Theorem There exists a unique function $\mu : \mathcal{P} \to \mathbb{R}$ (called the **area function**) that satisfies the following conditions:

- **(positivity)** $\mu(X) \geq 0$ for all $X \in \mathcal{P}$;
- **(additivity)** $\mu(X \cup Y) = \mu(X) + \mu(Y)$ if $X \cap Y = \emptyset$;
- **(translation invariance)** $\mu(X + \mathbf{v}) = \mu(X)$ for all $X \in \mathcal{P}$ and $\mathbf{v} \in \mathbb{R}^2$;
- $\mu(Q) = 1$, where $Q = [0,1] \times [0,1]$ is the unit square.

The area function satisfies an extra condition:

- **(monotonicity)** $\mu(X) \leq \mu(Y)$ whenever $X \subset Y$.

Now for any bounded set $X \subset \mathbb{R}^2$ we let $\overline{\mu}(X) = \inf\limits_{X \subset Y} \mu(Y)$ and $\underline{\mu}(X) = \sup\limits_{Z \subset X} \mu(Z)$. Note that $\underline{\mu}(X) \leq \overline{\mu}(X)$. In the case of equality, the set X is called **Jordan measurable** and we let $\mathrm{area}(X) = \overline{\mu}(X)$.

Area, volume, and determinants

- 2×2 determinants and plane geometry

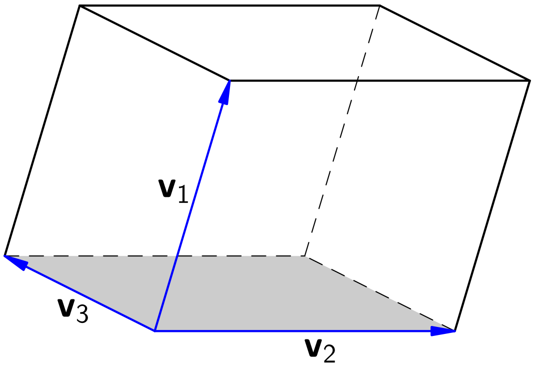
Let P be a parallelogram in the plane $\mathbb{R}^2$. Suppose that vectors $\mathbf{v}_1, \mathbf{v}_2 \in \mathbb{R}^2$ are represented by adjacent sides of P. Then $\mathrm{area}(P) = |\det A|$, where $A = (\mathbf{v}_1, \mathbf{v}_2)$, a matrix whose columns are $\mathbf{v}_1$ and $\mathbf{v}_2$.

Consider a linear operator $L_A : \mathbb{R}^2 \to \mathbb{R}^2$ given by $L_A(\mathbf{v}) = A\mathbf{v}$ for any column vector $\mathbf{v}$. Then $\mathrm{area}(L_A(D)) = |\det A| \, \mathrm{area}(D)$ for any bounded domain D.

- 3×3 determinants and space geometry

Let Π be a parallelepiped in space $\mathbb{R}^3$. Suppose that vectors $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3 \in \mathbb{R}^3$ are represented by adjacent edges of Π. Then $\mathrm{volume}(\Pi) = |\det B|$, where $B = (\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3)$, a matrix whose columns are $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_3$.

Similarly, $\mathrm{volume}(L_B(D)) = |\det B| \, \mathrm{volume}(D)$ for any bounded domain $D \subset \mathbb{R}^3$.



$\mathrm{volume}(\Pi) = |\det B|$, where $B = (\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3)$. Note that the parallelepiped Π is the image under L_B of a unit cube whose adjacent edges are $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$.

The triple $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ obeys the right-hand rule. We say that L_B **preserves orientation** if it preserves the hand rule for any basis. This is the case if and only if $\det B > 0$.

Riemann sums in two dimensions

Consider a closed coordinate rectangle
$R = [a, b] \times [c, d] \subset \mathbb{R}^2$.

Definition. A **Riemann sum** of a function $f : R \to \mathbb{R}$ with respect to a partition $P = \{D_1, D_2, \dots, D_n\}$ of R generated by samples $t_j \in D_j$ is a sum

$$\mathcal{S}(f, P, t_j) = \sum_{j=1}^{n} f(t_j)\, \mathrm{area}(D_j).$$

The norm of the partition P is $\|P\| = \max_{1 \le j \le n} \mathrm{diam}(D_j)$.

Definition. The Riemann sums $\mathcal{S}(f, P, t_j)$ **converge** to a limit $I(f)$ as the norm $\|P\| \to 0$ if for every $\varepsilon > 0$ there exists $\delta > 0$ such that $\|P\| < \delta$ implies $|\mathcal{S}(f, P, t_j) - I(f)| < \varepsilon$ for any partition P and choice of samples t_j.

If this is the case, then the function f is called **integrable** on R and the limit $I(f)$ is called the **integral** of f over R.

Double integral

Closed coordinate rectangle $R = [a, b] \times [c, d]$
$= \{(x, y) \in \mathbb{R}^2 \mid a \le x \le b,\ \ c \le y \le d\}$.

Notation: $\displaystyle\iint_R f\,dA$ or $\displaystyle\iint_R f(x, y)\,dx\,dy$.

Theorem 1 If f is continuous on the closed rectangle R, then f is integrable.

Theorem 2 A function $f : R \to \mathbb{R}$ is Riemann integrable on the rectangle R if and only if f is bounded on R and continuous almost everywhere on R (that is, the set of discontinuities of f has zero area).

Fubini's Theorem

Fubini's Theorem allows us to reduce a multiple integral to a repeated one-dimensional integral.

Theorem If a function f is integrable on $R = [a, b] \times [c, d]$, then

$$\iint_R f\, dA = \int_a^b \Big(\int_c^d f(x, y)\, dy \Big)\, dx$$
$$= \int_c^d \Big(\int_a^b f(x, y)\, dx \Big)\, dy.$$

In particular, this implies that we can change the order of integration in a repeated integral.

Integrals over general domains

Suppose $f : D \to \mathbb{R}$ is a function defined on a (Jordan) measurable set $D \subset \mathbb{R}^2$. Since D is bounded, it is contained in a rectangle R. To define the integral of f over D, we extend the function f to a function on R:

$$f^{\mathrm{ext}}(x, y) = \begin{cases} f(x, y) & \text{if } (x, y) \in D, \\ 0 & \text{if } (x, y) \notin D. \end{cases}$$

Definition. $\displaystyle\iint_D f \, dA$ is defined to be $\displaystyle\iint_R f^{\mathrm{ext}} \, dA$.

In particular, $\displaystyle \mathrm{area}(D) = \iint_D 1 \, dA$.

Integration as a linear operation

Theorem 1 If functions f, g are integrable on a set $D \subset \mathbb{R}^2$, then the sum $f + g$ is also integrable on D and

$$\iint_D (f+g)\, dA = \iint_D f\, dA + \iint_D g\, dA.$$

Theorem 2 If a function f is integrable on a set $D \subset \mathbb{R}^2$, then for each $\alpha \in \mathbb{R}$ the scalar multiple αf is also integrable on D and

$$\iint_D \alpha f\, dA = \alpha \iint_D f\, dA.$$

More properties of integrals

Theorem 3 If functions f, g are integrable on a set $D \subset \mathbb{R}^2$, and $f(x, y) \leq g(x, y)$ for all $(x, y) \in D$, then

$$\iint_D f\, dA \leq \iint_D g\, dA.$$

Theorem 4 If a function f is integrable on sets $D_1, D_2 \subset \mathbb{R}^2$, then it is integrable on their union $D_1 \cup D_2$. Moreover, if the sets D_1 and D_2 are disjoint up to a set of zero area, then

$$\iint_{D_1 \cup D_2} f\, dA = \iint_{D_1} f\, dA + \iint_{D_2} f\, dA.$$

Change of variables in a double integral

Theorem Let $D \subset \mathbb{R}^2$ be a measurable domain and f be an integrable function on D. If $\mathbf{T} = (u, v)$ is a smooth coordinate mapping such that $\mathbf{T}^{-1}$ is defined on D, Then

$$\iint_D f(u, v)\, du\, dv$$

$$= \iint_{\mathbf{T}^{-1}(D)} f\left(u(x, y), v(x, y)\right) \left|\det \frac{\partial(u, v)}{\partial(x, y)}\right| \, dx\, dy.$$

In particular, the integral in the right-hand side is well defined.

Problem Evaluate a double integral

$$\iint_P \big(2x + 3y - \cos(\pi x + 2\pi y)\big)\, dx\, dy$$

over a parallelogram P with vertices $(-1,-1)$, $(1,0)$, $(2,2)$, and $(0,1)$.

Adjacent edges of the parallelogram P are represented by vectors $\mathbf{v}_1 = (1,0) - (-1,-1) = (2,1)$ and $\mathbf{v}_2 = (0,1) - (-1,-1) = (1,2)$.

Consider a transformation L of the plane $\mathbb{R}^2$ given by

$$L\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}\begin{pmatrix} u \\ v \end{pmatrix} + \begin{pmatrix} -1 \\ -1 \end{pmatrix} = \begin{pmatrix} 2u + v - 1 \\ u + 2v - 1 \end{pmatrix}$$

(columns of the matrix are vectors $\mathbf{v}_1$ and $\mathbf{v}_2$). By construction, L maps the unit square $[0,1]\times[0,1]$ onto the parallelogram P. The Jacobian matrix J of L is the same at any point: $J = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$.

Changing coordinates in the integral from (x, y) to (u, v) so that $(x, y) = L(u, v) = (2u + v - 1,\ u + 2v - 1)$, we obtain

$$\iint_P \big(2x + 3y - \cos(\pi x + 2\pi y)\big)\, dx\, dy$$

$$= \iint_{L^{-1}(P)} \big(7u + 8v - 5 - \cos(4\pi u + 5\pi v - 3\pi)\big)\, |\det J|\, du\, dv$$

$$= \int_0^1 \int_0^1 3\big(7u + 8v - 5 + \cos(4\pi u + 5\pi v)\big)\, du\, dv$$

$$= \frac{21}{2} + 12 - 15 + \int_0^1 \int_0^1 3\cos(4\pi u + 5\pi v)\, du\, dv.$$

Further, $\displaystyle\int_0^1 3\cos(4\pi u + 5\pi v)\, du = \frac{3}{4\pi} \sin(4\pi u + 5\pi v)\,\Big|_{u=0}^{1}$

$$= \frac{3}{4\pi}\big(\sin(4\pi + 5\pi v) - \sin(5\pi v)\big) = 0 \text{ for all } v.$$

It follows that $\displaystyle\iint_P \big(2x + 3y - \cos(\pi x + 2\pi y)\big)\, dx\, dy = \frac{15}{2}.$

Triple integral

To integrate in $\mathbb{R}^3$, volumes are used instead of areas in $\mathbb{R}^2$. Instead of coordinate rectangles, basic sets are coordinate boxes (or bricks) $B = [a_1, b_1] \times [a_2, b_2] \times [a_3, b_3] \subset \mathbb{R}^3$. Then we can define an integral of a function f over a measurable set $D \subset \mathbb{R}^3$.

Notation: $\iiint_D f \, dV$ or $\iiint_D f(x, y, z) \, dx \, dy \, dz$.

The properties of triple integrals are completely analogous to those of double integrals. In particular, Fubini's Theorem is formulated as follows.

Theorem If a function f is integrable on a brick $B = [a_1, b_1] \times [a_2, b_2] \times [a_3, b_3] \subset \mathbb{R}^3$, then

$$\iiint_B f \, dV = \int_{a_1}^{b_1} \left(\int_{a_2}^{b_2} \left(\int_{a_3}^{b_3} f(x, y, z) \, dz \right) dy \right) dx.$$