

MATH 311

Topics in Applied Mathematics I

Lecture 37:
Review for Test 3.

Topics for Test 3

Vector analysis (Leon/Colley 8.1–8.4, 9.1–9.5, 10.1–10.3, 11.1–11.3)

- Gradient, divergence, and curl
- Fubini's Theorem
- Change of coordinates in a multiple integral
- Length of a curve
- Line integrals
- Green's Theorem
- Conservative vector fields
- Area of a surface
- Surface integrals
- Gauss' Theorem
- Stokes' Theorem

Sample problems for Test 3

Problem 1 Find $\text{curl}(\text{curl}(\mathbf{F}))$, where

$$\mathbf{F}(x, y, z) = (x^2 + y^2)\mathbf{e}_1 + ze^{x+y}\mathbf{e}_2 + (x + \sin y)\mathbf{e}_3.$$

Problem 2 Evaluate a double integral

$$\iint_P (2x + 3y - \cos(\pi x + 2\pi y)) \, dx \, dy$$

over a parallelogram P with vertices $(-1, -1)$, $(1, 0)$, $(2, 2)$, and $(0, 1)$.

Sample problems for Test 3

Problem 3 Find the area of a cardioid which boundary is given by $r = 1 - \cos \phi$ in polar coordinates.

Problem 4 Consider a vector field $\mathbf{F}(x, y, z) = (yz + 2 \cos 2x, xz - e^z, xy - ye^z)$.

- (i) Verify that the field \mathbf{F} is conservative.
- (ii) Find a function f such that $\mathbf{F} = \nabla f$.

Sample problems for Test 3

Problem 5 Let C be a solid cylinder bounded by planes $z = 0$, $z = 2$ and a cylindrical surface $x^2 + y^2 = 1$. Orient the boundary ∂C with outward normals and evaluate a surface integral

$$\oiint_{\partial C} (x^2 \mathbf{e}_1 + y^2 \mathbf{e}_2 + z^2 \mathbf{e}_3) \cdot d\mathbf{S}.$$

Problem 6 Let D be a region in \mathbb{R}^3 bounded by a paraboloid $z = x^2 + y^2$ and a plane $z = 9$. Let S denote the part of the paraboloid that bounds D , oriented by outward normals. Evaluate a surface integral

$$\iint_S \operatorname{curl}(\mathbf{F}) \cdot d\mathbf{S},$$

where $\mathbf{F}(x, y, z) = (e^{x^2+z^2}, xy + xz + yz, e^{xyz})$.

Problem 1 Find $\text{curl}(\text{curl}(\mathbf{F}))$, where

$$\mathbf{F}(x, y, z) = (x^2 + y^2)\mathbf{e}_1 + ze^{x+y}\mathbf{e}_2 + (x + \sin y)\mathbf{e}_3.$$

For any vector field $\mathbf{F} = (F_1, F_2, F_3)$ we have, informally,

$$\text{curl } \mathbf{F} = \nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{e}_1 & \mathbf{e}_2 & \mathbf{e}_3 \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_1 & F_2 & F_3 \end{vmatrix}$$

or, formally,

$$\text{curl } \mathbf{F} = \left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}, \frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x}, \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \right).$$

Problem 1 Find $\text{curl}(\text{curl}(\mathbf{F}))$, where

$$\mathbf{F}(x, y, z) = (x^2 + y^2)\mathbf{e}_1 + ze^{x+y}\mathbf{e}_2 + (x + \sin y)\mathbf{e}_3.$$

Let $\mathbf{G} = \text{curl } \mathbf{F}$, $\mathbf{G} = (G_1, G_2, G_3)$. We obtain

$$G_1 = \frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z} = \frac{\partial}{\partial y}(x + \sin y) - \frac{\partial}{\partial z}(ze^{x+y}) = \cos y - e^{x+y},$$

$$G_2 = \frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x} = \frac{\partial}{\partial z}(x^2 + y^2) - \frac{\partial}{\partial x}(x + \sin y) = -1,$$

$$G_3 = \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} = \frac{\partial}{\partial x}(ze^{x+y}) - \frac{\partial}{\partial y}(x^2 + y^2) = ze^{x+y} - 2y.$$

Hence $\mathbf{G} = \text{curl } \mathbf{F} = (\cos y - e^{x+y}, -1, ze^{x+y} - 2y)$.

Now let $\mathbf{H} = \text{curl } \mathbf{G}$, $\mathbf{H} = (H_1, H_2, H_3)$. We obtain

$$H_1 = \frac{\partial G_3}{\partial y} - \frac{\partial G_2}{\partial z} = \frac{\partial}{\partial y}(ze^{x+y} - 2y) - \frac{\partial}{\partial z}(-1) = ze^{x+y} - 2,$$

$$H_2 = \frac{\partial G_1}{\partial z} - \frac{\partial G_3}{\partial x} = \frac{\partial}{\partial z}(\cos y - e^{x+y}) - \frac{\partial}{\partial x}(ze^{x+y} - 2y) = -ze^{x+y},$$

$$H_3 = \frac{\partial G_2}{\partial x} - \frac{\partial G_1}{\partial y} = \frac{\partial}{\partial x}(-1) - \frac{\partial}{\partial y}(\cos y - e^{x+y}) = \sin y + e^{x+y}.$$

Thus $\text{curl}(\text{curl}(\mathbf{F})) = (ze^{x+y} - 2, -ze^{x+y}, \sin y + e^{x+y})$.

Problem 2 Evaluate a double integral

$$\iint_P (2x + 3y - \cos(\pi x + 2\pi y)) \, dx \, dy$$

over a parallelogram P with vertices $(-1, -1)$, $(1, 0)$, $(2, 2)$, and $(0, 1)$.

Let us change coordinates in this integral so that the domain of integration becomes the unit square $Q = [0, 1] \times [0, 1]$.

We are going to use a substitution of the form

$$(x, y) = L(u, v) = (a_{11}u + a_{12}v + b_1, a_{21}u + a_{22}v + b_2),$$

where a_{ij}, b_i are constants. The constants are determined from the conditions $L(0, 0) = (-1, -1)$, $L(1, 0) = (1, 0)$, and $L(0, 1) = (0, 1)$. We obtain that

$$L \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} 2u + v - 1 \\ u + 2v - 1 \end{pmatrix} = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} + \begin{pmatrix} -1 \\ -1 \end{pmatrix}.$$

The Jacobian matrix J of L is constant: $J = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$.

Changing coordinates in the integral from (x, y) to (u, v) so that $(x, y) = L(u, v) = (2u + v - 1, u + 2v - 1)$, we obtain

$$\begin{aligned} & \iint_P (2x + 3y - \cos(\pi x + 2\pi y)) \, dx \, dy \\ &= \iint_{L^{-1}(P)} (7u + 8v - 5 - \cos(4\pi u + 5\pi v - 3\pi)) |\det J| \, du \, dv \\ &= \int_0^1 \int_0^1 3(7u + 8v - 5 + \cos(4\pi u + 5\pi v)) \, du \, dv \\ &= \frac{21}{2} + 12 - 15 + \int_0^1 \int_0^1 3 \cos(4\pi u + 5\pi v) \, du \, dv. \end{aligned}$$

$$\begin{aligned} \text{Further, } & \int_0^1 3 \cos(4\pi u + 5\pi v) \, du = \frac{3}{4\pi} \sin(4\pi u + 5\pi v) \Big|_{u=0}^1 \\ &= \frac{3}{4\pi} (\sin(4\pi + 5\pi v) - \sin(5\pi v)) = 0 \text{ for all } v. \end{aligned}$$

$$\text{It follows that } \iint_P (2x + 3y - \cos(\pi x + 2\pi y)) \, dx \, dy = \frac{15}{2}.$$

Problem 3 Find the area of a cardioid which boundary is given by $r = 1 - \cos \phi$ in polar coordinates.

Let H denote the cardioid. The boundary ∂H is parametrized (in Cartesian coordinates) by a path $\mathbf{x} : [0, 2\pi] \rightarrow \mathbb{R}^2$, $\mathbf{x}(t) = ((1 - \cos t) \cos t, (1 - \cos t) \sin t)$. The cardioid is on the left when going along \mathbf{x} . By Green's Theorem,

$$\int_{\mathbf{x}} (-y \, dx + x \, dy) = \iint_H 2 \, dx \, dy = 2 \, \text{area}(H).$$

Evaluating the line integral, we will find the area of the cardioid.

Problem 4 Consider a vector field $\mathbf{F}(x, y, z) = (yz + 2 \cos 2x, xz - e^z, xy - ye^z)$.

(i) Verify that the field \mathbf{F} is conservative.

Since \mathbf{F} is a smooth vector field on the entire space, it is conservative if and only if its Jacobian matrix is symmetric everywhere in \mathbb{R}^3 . For vector fields on \mathbb{R}^3 , this is equivalent to $\text{curl}(\mathbf{F}) = \mathbf{0}$. We have to verify three identities.

$$\frac{\partial F_1}{\partial y} = \frac{\partial F_2}{\partial x}: \quad \frac{\partial}{\partial y}(yz + 2 \cos 2x) = \frac{\partial}{\partial x}(xz - e^z) \iff z = z,$$

$$\frac{\partial F_1}{\partial z} = \frac{\partial F_3}{\partial x}: \quad \frac{\partial}{\partial z}(yz + 2 \cos 2x) = \frac{\partial}{\partial x}(xy - ye^z) \iff y = y,$$

$$\frac{\partial F_2}{\partial z} = \frac{\partial F_3}{\partial y}: \quad \frac{\partial}{\partial z}(xz - e^z) = \frac{\partial}{\partial y}(xy - ye^z) \\ \iff x - e^z = x - e^z.$$

Problem 4 Consider a vector field

$$\mathbf{F}(x, y, z) = (yz + 2 \cos 2x, xz - e^z, xy - ye^z).$$

(ii) Find a function f such that $\mathbf{F} = \nabla f$.

We are looking for a function $f : \mathbb{R}^3 \rightarrow \mathbb{R}$ such that

$$\frac{\partial f}{\partial x} = yz + 2 \cos 2x, \quad \frac{\partial f}{\partial y} = xz - e^z, \quad \frac{\partial f}{\partial z} = xy - ye^z.$$

Integrating the third equality by z , we get

$$f(x, y, z) = \int (xy - ye^z) dz = xyz - ye^z + g(x, y).$$

Substituting this into the other equalities, we obtain that

$$yz + g'_x = yz + 2 \cos 2x \quad \text{and} \quad xz - e^z + g'_y = xz - e^z.$$

Hence $g'_y = 0$ so that g does not depend on y . Since

$$g'_x = 2 \cos 2x, \quad \text{we obtain that}$$

$$g(x, y) = \int 2 \cos 2x dx = \sin 2x + c, \quad \text{where } c \text{ is a constant.}$$

Finally, $f(x, y, z) = xyz - ye^z + \sin 2x + c$.

Problem 4 Consider a vector field
 $\mathbf{F}(x, y, z) = (yz + 2 \cos 2x, xz - e^z, xy - ye^z)$.
(ii) Find a function f such that $\mathbf{F} = \nabla f$.

Alternative solution: If $\mathbf{F} = \nabla f$, then

$$\int_{\mathbf{x}} \mathbf{F} \cdot d\mathbf{s} = f(B) - f(A)$$

for any points $A, B \in \mathbb{R}^3$ and any path \mathbf{x} joining A to B .
We can use this relation to recover the function f .

For any given point $A = (x, y, z)$ we consider a linear path \mathbf{x}_A from the origin to A , $\mathbf{x}_A : [0, 1] \rightarrow \mathbb{R}^3$, $\mathbf{x}_A(t) = (tx, ty, tz)$.
Then

$$f(A) - f(\mathbf{0}) = \int_{\mathbf{x}_A} \mathbf{F} \cdot d\mathbf{s} = \int_0^1 \mathbf{F}(\mathbf{x}_A(t)) \cdot \mathbf{x}'_A(t) dt.$$

$$\begin{aligned}
f(A) - f(\mathbf{0}) &= \int_{\mathbf{x}_A} \mathbf{F} \cdot d\mathbf{s} = \int_0^1 \mathbf{F}(\mathbf{x}_A(t)) \cdot \mathbf{x}'_A(t) dt \\
&= \int_0^1 (t^2yz + 2 \cos 2tx, t^2xz - e^{tz}, t^2xy - tye^{tz}) \cdot (x, y, z) dt \\
&= \int_0^1 ((t^2yz + 2 \cos 2tx)x + (t^2xz - e^{tz})y + (t^2xy - tye^{tz})z) dt \\
&= \int_0^1 (3t^2xyz + 2x \cos 2tx - ye^{tz} - tye^{tz}) dt \\
&= t^3xyz \Big|_{t=0}^1 + \sin 2tx \Big|_{t=0}^1 - yte^{tz} \Big|_{t=0}^1 = xyz + \sin 2x - ye^z.
\end{aligned}$$

Thus $f(x, y, z) = xyz + \sin 2x - ye^z + c$, where $c = f(\mathbf{0})$ is a constant.

Problem 5 Let C be a solid cylinder bounded by planes $z = 0$, $z = 2$ and a cylindrical surface $x^2 + y^2 = 1$. Orient the boundary ∂C with outward normals and evaluate a surface integral

$$\iint_{\partial C} (x^2 \mathbf{e}_1 + y^2 \mathbf{e}_2 + z^2 \mathbf{e}_3) \cdot d\mathbf{S}.$$

By Gauss' Theorem,

$$\begin{aligned} \iint_{\partial C} (x^2 \mathbf{e}_1 + y^2 \mathbf{e}_2 + z^2 \mathbf{e}_3) \cdot d\mathbf{S} &= \iiint_C \nabla \cdot (x^2 \mathbf{e}_1 + y^2 \mathbf{e}_2 + z^2 \mathbf{e}_3) dV \\ &= \iiint_C \left(\frac{\partial}{\partial x}(x^2) + \frac{\partial}{\partial y}(y^2) + \frac{\partial}{\partial z}(z^2) \right) dx dy dz \\ &= \iiint_C 2(x + y + z) dx dy dz. \end{aligned}$$

Consider an invertible linear transformation $L : \mathbb{R}^3 \rightarrow \mathbb{R}^3$ given by $L(x, y, z) = (-x, -y, z)$. The matrix of L (relative to the standard basis) is

$$M = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

It is also the Jacobian matrix of L at every point. Changing coordinates from (x, y, z) to (u, v, w) so that $(x, y, z) = L(u, v, w)$, we obtain

$$\begin{aligned} \iiint_C 2(x + y) \, dx \, dy \, dz &= \iiint_{L^{-1}(C)} 2(-u - v) |\det M| \, du \, dv \, dw \\ &= - \iiint_C 2(u + v) \, du \, dv \, dw. \end{aligned}$$

It follows that $\iiint_C 2(x + y) \, dx \, dy \, dz = 0$.

By linearity of the integral,

$$\begin{aligned} \oiint_{\partial C} (x^2 \mathbf{e}_1 + y^2 \mathbf{e}_2 + z^2 \mathbf{e}_3) \cdot d\mathbf{S} &= \iiint_C 2(x + y + z) \, dx \, dy \, dz \\ &= \iiint_C 2(x + y) \, dx \, dy \, dz + \iiint_C 2z \, dx \, dy \, dz = \iiint_C 2z \, dx \, dy \, dz. \end{aligned}$$

The cylinder C can be represented as $C = U \times [0, 2]$, where U is the unit disc in the plane,

$$U = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 \leq 1\}.$$

By Fubini's Theorem,

$$\begin{aligned} \iiint_C 2z \, dx \, dy \, dz &= \iint_U \left(\int_0^2 2z \, dz \right) \, dx \, dy \\ &= \iint_U 4 \, dx \, dy = 4 \operatorname{area}(U) = 4\pi. \end{aligned}$$

Problem 6 Let D be a region in \mathbb{R}^3 bounded by a paraboloid $z = x^2 + y^2$ and a plane $z = 9$. Let S denote the part of the paraboloid that bounds D , oriented by outward normals.

Evaluate a surface integral

$$\iint_S \operatorname{curl}(\mathbf{F}) \cdot d\mathbf{S},$$

where $\mathbf{F}(x, y, z) = (e^{x^2+z^2}, xy + xz + yz, e^{xyz})$.

$$\begin{aligned} \text{We have } \operatorname{curl} \mathbf{F} &= \left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}, \frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x}, \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \right) \\ &= (xze^{xyz} - x - y, 2ze^{x^2+z^2} - yze^{xyz}, y + z). \end{aligned}$$

Direct evaluation of the surface integral seems problematic. By Stokes' Theorem, the surface integral equals the integral of the field \mathbf{F} along the circle ∂S . However evaluation of this line integral seems problematic as well.

By the corollary of Stokes' Theorem,

$$\iint_{\partial D} \operatorname{curl}(\mathbf{F}) \cdot d\mathbf{S} = 0.$$

It follows that

$$\iint_S \operatorname{curl}(\mathbf{F}) \cdot d\mathbf{S} = - \iint_{\partial D \setminus S} \operatorname{curl}(\mathbf{F}) \cdot d\mathbf{S}.$$

We observe that $\partial D \setminus S$ is a horizontal disc $Q \times \{9\}$, where $Q = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 \leq 9\}$. It is oriented by the upward normal vector $\mathbf{n} = (0, 0, 1)$. Now

$$\begin{aligned} \iint_{\partial D \setminus S} \operatorname{curl}(\mathbf{F}) \cdot d\mathbf{S} &= \iint_{\partial D \setminus S} \operatorname{curl}(\mathbf{F}) \cdot \mathbf{n} \, dS \\ &= \iint_Q (y + 9) \, dx \, dy = \iint_Q 9 \, dx \, dy = 9 \operatorname{area}(Q) = 81\pi. \end{aligned}$$

Thus $\iint_S \operatorname{curl}(\mathbf{F}) \cdot d\mathbf{S} = -81\pi$.