MATH 311

Topics in Applied Mathematics I

Lecture 39:
Integration of differential forms.

Vector line and surface integrals

Any vector integral along a curve $\gamma \subset \mathbb{R}^n$ can be represented as a scalar line integral:

$$\int_\gamma \mathbf{F} \cdot d\mathbf{s} = \int_\gamma (\mathbf{F} \cdot \mathbf{t})\, ds,$$

where $\mathbf{t}$ is a unit **tangent** vector chosen according to the orientation of the curve γ.

Any vector integral along a surface $S \subset \mathbb{R}^3$ can be represented as a scalar surface integral:

$$\iint_S \mathbf{F} \cdot d\mathbf{S} = \iint_S (\mathbf{F} \cdot \mathbf{n})\, dS,$$

where $\mathbf{n}$ is a unit **normal** vector chosen according to the orientation of the surface S.

k-forms

Let V be a vector space. Given an integer $k \geq 0$, a k-**form** on V is a function $\omega : V^k \to \mathbb{R}$ such that

- ω is **multi-linear**, which means that it depends linearly on each of its k arguments; and
- ω is **anti-symmetric**, which means that its value changes the sign upon exchanging any two of the k arguments.

In particular, a 0-form is just a constant, a 1-form is merely a linear functional on V, and a 2-form is a bi-linear function $\omega : V \times V \to \mathbb{R}$ such that $\omega(\mathbf{v}, \mathbf{u}) = -\omega(\mathbf{u}, \mathbf{v})$ for all $\mathbf{v}, \mathbf{u} \in V$.

Principal example. For any vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n \in \mathbb{R}^n$ let $\omega(\mathbf{v}_1, \ldots, \mathbf{v}_n) = \det A$, where $A = (\mathbf{v}_1, \ldots, \mathbf{v}_n)$ is an $n \times n$ matrix whose consecutive columns are vectors $\mathbf{v}_1, \ldots, \mathbf{v}_n$. Then ω is an n-form on $\mathbb{R}^n$ (called the **volume form**).

Wedge product

Suppose $\omega_1, \omega_2, \ldots, \omega_k$ are linear functionals on a vector space V. The **wedge product** of these 1-forms, denoted $\omega_1 \wedge \omega_2 \wedge \cdots \wedge \omega_k$, is a k-form on V defined by

$$\omega_1 \wedge \cdots \wedge \omega_k(\mathbf{v}_1, \ldots, \mathbf{v}_k) = \begin{vmatrix} \omega_1(\mathbf{v}_1) & \omega_1(\mathbf{v}_2) & \cdots & \omega_1(\mathbf{v}_k) \\ \omega_2(\mathbf{v}_1) & \omega_2(\mathbf{v}_2) & \cdots & \omega_2(\mathbf{v}_k) \\ \vdots & \vdots & \ddots & \vdots \\ \omega_k(\mathbf{v}_1) & \omega_k(\mathbf{v}_2) & \cdots & \omega_k(\mathbf{v}_k) \end{vmatrix}.$$

Note that dependence of the wedge product $\omega_1 \wedge \omega_2 \wedge \cdots \wedge \omega_k$ on its factors is also multi-linear and anti-symmetric.

Now suppose $V = \mathbb{R}^n$. Let ξ_i denote a linear functional on $\mathbb{R}^n$ that evaluates the i-th coordinate for each vector. Then the volume form from the previous slide is $\xi_1 \wedge \xi_2 \wedge \cdots \wedge \xi_n$. The set of all k-forms on $\mathbb{R}^n$, denoted $\Lambda^k(\mathbb{R}^n)^*$, is a vector space. It has a basis comprised of wedge products $\xi_{i_1} \wedge \xi_{i_2} \wedge \cdots \wedge \xi_{i_k}$, where $1 \leq i_1 < i_2 < \cdots < i_k \leq n$.

Differential k-forms

Let $U \subset \mathbb{R}^n$ be an open region. A **differential k-form** on U is a field of k-forms from $\Lambda^k(\mathbb{R}^n)^*$. Formally, its a mapping $\omega : U \to \Lambda^k(\mathbb{R}^n)^*$.

Example. Consider a smooth function $f : U \to \mathbb{R}$ (which is an example of a differential 0-form). To each point $p \in U$ we assign a linear functional $\mathbf{v} \mapsto D_{\mathbf{v}} f(p)$ (the derivative of f at p). This defines a differential 1-form, which is denoted df.

Let $x_1, x_2, \ldots, x_n$ be coordinates in $\mathbb{R}^n$. Each x_i can be regarded a smooth function on U. Note that dx_i is a constant field: its value is ξ_i at every point. It follows that any differential k-form ω on U is uniquely represented as

$$\omega = \sum_{1 \le i_1 < i_2 < \cdots < i_k \le n} \alpha_{i_1 i_2 \ldots i_k} \, dx_{i_1} \wedge dx_{i_2} \wedge \cdots \wedge dx_{i_k},$$

where $\alpha_{i_1 i_2 \ldots i_k}$ are some functions on U and the wedge product is pointwise. The form ω is **smooth** if each $\alpha_{i_1 i_2 \ldots i_k}$ is smooth.

Integration of differential forms

Any continuous differential k-form ω in a region $U \subset \mathbb{R}^n$ can be integrated over a smooth oriented k-**dimensional manifold** in U.

Definition. Let $R \subset \mathbb{R}^k$ be a connected, bounded region. A continuous one-to-one map $\mathbf{X} : R \to \mathbb{R}^n$ is called a **parametrized** k-**dimensional manifold**. The parametrized manifold is **smooth** if $\mathbf{X}$ is smooth and, moreover, the Jacobian matrix of $\mathbf{X}$ has rank k at every point of R.

If
$$\omega = \sum_{1\le i_1<i_2<\dots<i_k\le n} \alpha_{i_1 i_2 \dots i_k}\, dx_{i_1} \wedge dx_{i_2} \wedge \dots \wedge dx_{i_k},$$
then
$$\int_{\mathbf{X}} \omega = \sum \int_R \alpha_{i_1 i_2 \dots i_k}(\mathbf{X}(s_1,\dots,s_k)) \det \frac{\partial(X_{i_1},\dots,X_{i_k})}{\partial(s_1,\dots,s_k)}\, dV.$$

Examples in $\mathbb{R}^3$. • Vector line integral

The integral of a vector field $\mathbf{F} = (F_1, F_2, F_3)$ along a curve γ can be interpreted as the integral of a differential 1-form:

$$\int_\gamma \mathbf{F} \cdot d\mathbf{s} = \int_\gamma F_1\, dx + F_2\, dy + F_3\, dz.$$

- Vector surface integral

The integral of a vector field $\mathbf{F} = (F_1, F_2, F_3)$ along a surface S can be interpreted as the integral of a differential 2-form:

$$\iint_S \mathbf{F} \cdot d\mathbf{S} = \iint_S F_1\, dy \wedge dz + F_2\, dz \wedge dx + F_3\, dx \wedge dy.$$

- Multiple integral

The integral of a function f over a region $U \subset \mathbb{R}^3$ can be interpreted as the integral of a differential 3-form:

$$\iiint_U f\, dV = \iiint_U f\, dx \wedge dy \wedge dz.$$

Exterior derivative

Let $U \subset \mathbb{R}^n$ be an open region. The vector space of differential k-forms on U is denoted $\Omega^k(U)$.

Theorem There exists a unique family of transformations $\delta_k : \Omega^k(U) \to \Omega^{k+1}(U)$, $k = 0, 1, 2, \ldots,$ such that

- each δ_k is linear,
- for any smooth function f on U, $\delta_0(f) = df$,
- for any smooth functions $f, g_1, \ldots, g_k$ on U,

$\delta_k(f\, dg_1 \wedge \cdots \wedge dg_k) = df \wedge dg_1 \wedge \cdots \wedge dg_k$.

The differential form $\delta_k(\omega)$ is called the **exterior derivative** of ω and denoted $d\omega$.

Generalized Stokes' Theorem For any smooth differential k-form ω on U and any bounded, oriented smooth $(k+1)$-dimensional manifold $C \subset U$,

$$\int_C d\omega = \oint_{\partial C} \omega.$$

Examples

- Differential 1-form in $\mathbb{R}^2$.

We have $\omega = M\,dx + N\,dy$. Then

$$\begin{aligned} d\omega &= d(M\,dx) + d(N\,dy) = dM \wedge dx + dN \wedge dy \\ &= \left(\frac{\partial M}{\partial x}dx + \frac{\partial M}{\partial y}dy\right) \wedge dx + \left(\frac{\partial N}{\partial x}dx + \frac{\partial N}{\partial y}dy\right) \wedge dy \\ &= \frac{\partial M}{\partial x}dx \wedge dx + \frac{\partial M}{\partial y}dy \wedge dx + \frac{\partial N}{\partial x}dx \wedge dy + \frac{\partial N}{\partial y}dy \wedge dy \\ &= \frac{\partial M}{\partial y}dy \wedge dx + \frac{\partial N}{\partial x}dx \wedge dy = \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right)dx \wedge dy. \end{aligned}$$

Hence in this case Generalized Stokes' Theorem yields Green's Theorem:

$$\oint_{\partial D} M\,dx + N\,dy = \iint_D \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy.$$

Examples

- Differential 1-form in $\mathbb{R}^3$.

We have $\omega = F_1\,dx + F_2\,dy + F_3\,dz$. Then

$$d\omega = \left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}\right) dy \wedge dz + \left(\frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x}\right) dz \wedge dx + \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right) dx \wedge dy.$$

In this case Generalized Stokes' Theorem yields usual Stokes' Theorem.

- Differential 2-form in $\mathbb{R}^3$.

We have $\omega = F_1\,dy \wedge dz + F_2\,dz \wedge dx + F_3\,dx \wedge dy$. Then

$$d\omega = \left(\frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}\right) dx \wedge dy \wedge dz.$$

In this case Generalized Stokes' Theorem yields Gauss' Theorem.