

MATH 311

Topics in Applied Mathematics I

Lecture 6:
Inverse matrix.

Identity matrix

Definition. The **identity matrix** (or **unit matrix**) is a diagonal matrix with all diagonal entries equal to 1. The $n \times n$ identity matrix is denoted I_n or simply I .

$$I_1 = (1), \quad I_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad I_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

In general,
$$I = \begin{pmatrix} 1 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 1 \end{pmatrix}.$$

Theorem. Let A be an arbitrary $m \times n$ matrix. Then $I_m A = A I_n = A$.

Inverse matrix

Let $\mathcal{M}_n(\mathbb{R})$ denote the set of all $n \times n$ matrices with real entries. We can **add**, **subtract**, and **multiply** elements of $\mathcal{M}_n(\mathbb{R})$. What about **division**?

Definition. Let $A \in \mathcal{M}_n(\mathbb{R})$. Suppose there exists an $n \times n$ matrix B such that

$$AB = BA = I_n.$$

Then the matrix A is called **invertible** and B is called the **inverse** of A (denoted A^{-1}).

A non-invertible square matrix is called **singular**.

$$\boxed{AA^{-1} = A^{-1}A = I}$$

Examples

$$A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}, \quad C = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}.$$

$$AB = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$

$$BA = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$

$$C^2 = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Thus $A^{-1} = B$, $B^{-1} = A$, and $C^{-1} = C$.

Basic properties of inverse matrices

- If $B = A^{-1}$ then $A = B^{-1}$. In other words, if A is invertible, so is A^{-1} , and $A = (A^{-1})^{-1}$.

- The inverse matrix (if it exists) is unique.

Moreover, if $AB = CA = I$ for some $n \times n$ matrices B and C , then $B = C = A^{-1}$.

Indeed, $B = IB = (CA)B = C(AB) = CI = C$.

- If $n \times n$ matrices A and B are invertible, so is AB , and $(AB)^{-1} = B^{-1}A^{-1}$.

$$(B^{-1}A^{-1})(AB) = B^{-1}(A^{-1}A)B = B^{-1}IB = B^{-1}B = I,$$
$$(AB)(B^{-1}A^{-1}) = A(BB^{-1})A^{-1} = AIA^{-1} = AA^{-1} = I.$$

- Similarly, $(A_1A_2 \dots A_k)^{-1} = A_k^{-1} \dots A_2^{-1}A_1^{-1}$.

Inverting diagonal matrices

Theorem A diagonal matrix $D = \text{diag}(d_1, \dots, d_n)$ is invertible if and only if all diagonal entries are nonzero: $d_i \neq 0$ for $1 \leq i \leq n$.

If D is invertible then $D^{-1} = \text{diag}(d_1^{-1}, \dots, d_n^{-1})$.

$$\begin{pmatrix} d_1 & 0 & \dots & 0 \\ 0 & d_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & d_n \end{pmatrix}^{-1} = \begin{pmatrix} d_1^{-1} & 0 & \dots & 0 \\ 0 & d_2^{-1} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & d_n^{-1} \end{pmatrix}$$

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If D is invertible then $D^{-1} = \text{diag}(d_1^{-1}, \dots, d_n^{-1})$.

Proof: If all $d_i \neq 0$ then, clearly,

$$\text{diag}(d_1, \dots, d_n) \text{diag}(d_1^{-1}, \dots, d_n^{-1}) = \text{diag}(1, \dots, 1) = I,$$

$$\text{diag}(d_1^{-1}, \dots, d_n^{-1}) \text{diag}(d_1, \dots, d_n) = \text{diag}(1, \dots, 1) = I.$$

Now suppose that $d_i = 0$ for some i . Then for any $n \times n$ matrix B the i th row of the matrix DB is a zero row. Hence $DB \neq I$.

Inverting 2×2 matrices

Definition. The **determinant** of a 2×2 matrix

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \text{ is } \det A = ad - bc.$$

Theorem A matrix $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is invertible if and only if $\det A \neq 0$.

If $\det A \neq 0$ then

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} = \frac{1}{ad - bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}.$$

Theorem A matrix $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is invertible if

and only if $\det A \neq 0$. If $\det A \neq 0$ then

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} = \frac{1}{ad - bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}.$$

Proof: Let $B = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$. Then

$$AB = BA = \begin{pmatrix} ad - bc & 0 \\ 0 & ad - bc \end{pmatrix} = (ad - bc)I_2.$$

In the case $\det A \neq 0$, we have $A^{-1} = (\det A)^{-1}B$.

In the case $\det A = 0$, the matrix A is not invertible as

$$\text{otherwise } AB = O \implies A^{-1}(AB) = A^{-1}O = O$$

$$\implies (A^{-1}A)B = O \implies I_2B = O \implies B = O$$

$$\implies A = O, \text{ but the zero matrix is singular.}$$

Problem. Solve a system $\begin{cases} 4x + 3y = 5, \\ 3x + 2y = -1. \end{cases}$

This system is equivalent to a matrix equation $A\mathbf{x} = \mathbf{b}$,
where $A = \begin{pmatrix} 4 & 3 \\ 3 & 2 \end{pmatrix}$, $\mathbf{x} = \begin{pmatrix} x \\ y \end{pmatrix}$, $\mathbf{b} = \begin{pmatrix} 5 \\ -1 \end{pmatrix}$.

We have $\det A = -1 \neq 0$. Hence A is invertible.

$$\begin{aligned} A\mathbf{x} = \mathbf{b} &\implies A^{-1}(A\mathbf{x}) = A^{-1}\mathbf{b} \implies (A^{-1}A)\mathbf{x} = A^{-1}\mathbf{b} \\ &\implies \mathbf{x} = A^{-1}\mathbf{b}. \end{aligned}$$

Conversely, $\mathbf{x} = A^{-1}\mathbf{b} \implies A\mathbf{x} = A(A^{-1}\mathbf{b}) = (AA^{-1})\mathbf{b} = \mathbf{b}$.

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 4 & 3 \\ 3 & 2 \end{pmatrix}^{-1} \begin{pmatrix} 5 \\ -1 \end{pmatrix} = \frac{1}{-1} \begin{pmatrix} 2 & -3 \\ -3 & 4 \end{pmatrix} \begin{pmatrix} 5 \\ -1 \end{pmatrix} = \begin{pmatrix} -13 \\ 19 \end{pmatrix}$$

System of n linear equations in n variables:

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2 \\ \dots\dots\dots \\ a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n = b_n \end{cases} \iff \mathbf{Ax} = \mathbf{b},$$

where

$$A = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{pmatrix}, \quad \mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}, \quad \mathbf{b} = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix}.$$

Theorem If the matrix A is invertible then the system has a unique solution, which is $\mathbf{x} = A^{-1}\mathbf{b}$.

General results on inverse matrices

Theorem 1 Given an $n \times n$ matrix A , the following conditions are equivalent:

- (i) A is invertible;
- (ii) $\mathbf{x} = \mathbf{0}$ is the only solution of the matrix equation $A\mathbf{x} = \mathbf{0}$;
- (iii) the matrix equation $A\mathbf{x} = \mathbf{b}$ has a unique solution for any n -dimensional column vector \mathbf{b} ;
- (iv) the row echelon form of A has no zero rows;
- (v) the reduced row echelon form of A is the identity matrix.

Theorem 2 Suppose that a sequence of elementary row operations converts a matrix A into the identity matrix.

Then the same sequence of operations converts the identity matrix into the inverse matrix A^{-1} .

Row echelon form of a square matrix:

$$\begin{pmatrix} \square & * & * & * & * & * & * \\ & \square & * & * & * & * & * \\ & & \square & * & * & * & * \\ & & & \square & * & * & * \\ & & & & \square & * & * \\ & & & & & \square & * \\ & & & & & & \square \end{pmatrix}$$

invertible case

$$\begin{pmatrix} \square & * & * & * & * & * & * \\ & \square & * & * & * & * & * \\ & & \square & * & * & * & * \\ & & & \square & * & * & * \\ & & & & \square & * & * \\ & & & & & \square & * \\ & & & & & & \square \end{pmatrix}$$

noninvertible case