

MATH 311

Topics in Applied Mathematics I

Lecture 7:

Inverse matrix (continued).

Transpose of a matrix.

Inverse matrix

Definition. Let A be an $n \times n$ matrix. The **inverse** of A is an $n \times n$ matrix, denoted A^{-1} , such that

$$\boxed{AA^{-1} = A^{-1}A = I.}$$

If A^{-1} exists then the matrix A is called **invertible**.
Otherwise A is called **singular**.

System of n linear equations in n variables:

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2 \\ \dots\dots\dots \\ a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n = b_n \end{cases} \iff \mathbf{Ax} = \mathbf{b},$$

where

$$A = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{pmatrix}, \quad \mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}, \quad \mathbf{b} = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix}.$$

Theorem If the matrix A is invertible then the system has a unique solution, which is $\mathbf{x} = A^{-1}\mathbf{b}$.

General results on inverse matrices

Theorem 1 Given an $n \times n$ matrix A , the following conditions are equivalent:

- (i) A is invertible;
- (ii) $\mathbf{x} = \mathbf{0}$ is the only solution of the matrix equation $A\mathbf{x} = \mathbf{0}$;
- (iii) the matrix equation $A\mathbf{x} = \mathbf{b}$ has a unique solution for any n -dimensional column vector \mathbf{b} ;
- (iv) the row echelon form of A has no zero rows;
- (v) the reduced row echelon form of A is the identity matrix.

Theorem 2 Suppose that a sequence of elementary row operations converts a matrix A into the identity matrix.

Then the same sequence of operations converts the identity matrix into the inverse matrix A^{-1} .

Row echelon form of a square matrix:

A 7x7 matrix in row echelon form. The diagonal elements are represented by squares, and the entries to the right of each square are asterisks. A blue line connects the bottom-right corners of the squares, forming a staircase pattern that descends from the top-left to the bottom-right.

invertible case

A 7x7 matrix in row echelon form, similar to the first one. However, the asterisks in the third row, fourth column and the sixth row, seventh column are circled in red. A blue line connects the bottom-right corners of the squares, forming a staircase pattern that descends from the top-left to the bottom-right.

noninvertible case

For any matrix in row echelon form, the number of columns with leading entries equals the number of rows with leading entries. For a square matrix, also the number of columns *without* leading entries (i.e., the number of free variables in a related system of linear equations) equals the number of rows *without* leading entries (i.e., zero rows).

Row echelon form of a square matrix:

$$\begin{pmatrix} \square & * & * & * & * & * & * \\ & \square & * & * & * & * & * \\ & & \square & * & * & * & * \\ & & & \square & * & * & * \\ & & & & \square & * & * \\ & & & & & \square & * \\ & & & & & & \square \end{pmatrix}$$

invertible case

$$\begin{pmatrix} \square & * & * & * & * & * & * \\ & \square & * & * & * & * & * \\ & & \square & * & * & * & * \\ & & & \square & * & * & * \\ & & & & \square & * & * \\ & & & & & \square & * \\ & & & & & & \square \end{pmatrix}$$

noninvertible case

Hence the row echelon form of a square matrix A is either strict triangular or else it has a zero row. In the former case, the equation $A\mathbf{x} = \mathbf{b}$ *always* has a unique solution. In the latter case, $A\mathbf{x} = \mathbf{b}$ *never* has a unique solution. Also, in the former case the reduced row echelon form of A is I .

Example. $A = \begin{pmatrix} 3 & -2 & 0 \\ 1 & 0 & 1 \\ -2 & 3 & 0 \end{pmatrix}.$

To check whether A is invertible, we convert it to row echelon form.

Interchange the 1st row with the 2nd row:

$$\begin{pmatrix} 1 & 0 & 1 \\ 3 & -2 & 0 \\ -2 & 3 & 0 \end{pmatrix}$$

Add -3 times the 1st row to the 2nd row:

$$\begin{pmatrix} 1 & 0 & 1 \\ 0 & -2 & -3 \\ -2 & 3 & 0 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 & 1 \\ 0 & -2 & -3 \\ -2 & 3 & 0 \end{pmatrix}$$

Add 2 times the 1st row to the 3rd row:

$$\begin{pmatrix} 1 & 0 & 1 \\ 0 & -2 & -3 \\ 0 & 3 & 2 \end{pmatrix}$$

Multiply the 2nd row by -0.5 :

$$\begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1.5 \\ 0 & 3 & 2 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1.5 \\ 0 & 3 & 2 \end{pmatrix}$$

Add -3 times the 2nd row to the 3rd row:

$$\begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1.5 \\ 0 & 0 & -2.5 \end{pmatrix}$$

Multiply the 3rd row by -0.4 :

$$\begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1.5 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} \boxed{1} & 0 & 1 \\ 0 & \boxed{1} & 1.5 \\ 0 & 0 & \boxed{1} \end{pmatrix}$$

We already know that the matrix A is invertible.

Let's proceed towards reduced row echelon form.

Add -1.5 times the 3rd row to the 2nd row:

$$\begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Add -1 times the 3rd row to the 1st row:

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

To obtain A^{-1} , we need to apply the following sequence of elementary row operations to the identity matrix:

- interchange the 1st row with the 2nd row,
- add -3 times the 1st row to the 2nd row,
- add 2 times the 1st row to the 3rd row,
- multiply the 2nd row by -0.5 ,
- add -3 times the 2nd row to the 3rd row,
- multiply the 3rd row by -0.4 ,
- add -1.5 times the 3rd row to the 2nd row,
- add -1 times the 3rd row to the 1st row.

A convenient way to compute the inverse matrix A^{-1} is to merge the matrices A and I into one 3×6 matrix $(A | I)$, and apply elementary row operations to this new matrix.

$$A = \begin{pmatrix} 3 & -2 & 0 \\ 1 & 0 & 1 \\ -2 & 3 & 0 \end{pmatrix}, \quad I = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$(A | I) = \left(\begin{array}{ccc|ccc} 3 & -2 & 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ -2 & 3 & 0 & 0 & 0 & 1 \end{array} \right)$$

$$\left(\begin{array}{ccc|ccc} 3 & -2 & 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ -2 & 3 & 0 & 0 & 0 & 1 \end{array} \right)$$

Interchange the 1st row with the 2nd row:

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 1 & 0 & 1 & 0 \\ 3 & -2 & 0 & 1 & 0 & 0 \\ -2 & 3 & 0 & 0 & 0 & 1 \end{array} \right)$$

Add -3 times the 1st row to the 2nd row:

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & -2 & -3 & 1 & -3 & 0 \\ -2 & 3 & 0 & 0 & 0 & 1 \end{array} \right)$$

$$\left(\begin{array}{ccc|cc} 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & -2 & -3 & 1 & -3 & 0 \\ -2 & 3 & 0 & 0 & 0 & 1 \end{array} \right)$$

Add 2 times the 1st row to the 3rd row:

$$\left(\begin{array}{ccc|cc} 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & -2 & -3 & 1 & -3 & 0 \\ 0 & 3 & 2 & 0 & 2 & 1 \end{array} \right)$$

Multiply the 2nd row by -0.5 :

$$\left(\begin{array}{ccc|cc} 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1.5 & -0.5 & 1.5 & 0 \\ 0 & 3 & 2 & 0 & 2 & 1 \end{array} \right)$$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1.5 & -0.5 & 1.5 & 0 \\ 0 & 3 & 2 & 0 & 2 & 1 \end{array} \right)$$

Add -3 times the 2nd row to the 3rd row:

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1.5 & -0.5 & 1.5 & 0 \\ 0 & 0 & -2.5 & 1.5 & -2.5 & 1 \end{array} \right)$$

Multiply the 3rd row by -0.4 :

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1.5 & -0.5 & 1.5 & 0 \\ 0 & 0 & 1 & -0.6 & 1 & -0.4 \end{array} \right)$$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1.5 & -0.5 & 1.5 & 0 \\ 0 & 0 & 1 & -0.6 & 1 & -0.4 \end{array} \right)$$

Add -1.5 times the 3rd row to the 2nd row:

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0.4 & 0 & 0.6 \\ 0 & 0 & 1 & -0.6 & 1 & -0.4 \end{array} \right)$$

Add -1 times the 3rd row to the 1st row:

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 0.6 & 0 & 0.4 \\ 0 & 1 & 0 & 0.4 & 0 & 0.6 \\ 0 & 0 & 1 & -0.6 & 1 & -0.4 \end{array} \right) = (I | A^{-1})$$

Thus
$$\begin{pmatrix} 3 & -2 & 0 \\ 1 & 0 & 1 \\ -2 & 3 & 0 \end{pmatrix}^{-1} = \begin{pmatrix} \frac{3}{5} & 0 & \frac{2}{5} \\ \frac{2}{5} & 0 & \frac{3}{5} \\ -\frac{3}{5} & 1 & -\frac{2}{5} \end{pmatrix}.$$

That is,

$$\begin{pmatrix} 3 & -2 & 0 \\ 1 & 0 & 1 \\ -2 & 3 & 0 \end{pmatrix} \begin{pmatrix} \frac{3}{5} & 0 & \frac{2}{5} \\ \frac{2}{5} & 0 & \frac{3}{5} \\ -\frac{3}{5} & 1 & -\frac{2}{5} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

$$\begin{pmatrix} \frac{3}{5} & 0 & \frac{2}{5} \\ \frac{2}{5} & 0 & \frac{3}{5} \\ -\frac{3}{5} & 1 & -\frac{2}{5} \end{pmatrix} \begin{pmatrix} 3 & -2 & 0 \\ 1 & 0 & 1 \\ -2 & 3 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Why does it work?

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{pmatrix} = \begin{pmatrix} a_1 & a_2 & a_3 \\ 2b_1 & 2b_2 & 2b_3 \\ c_1 & c_2 & c_3 \end{pmatrix},$$

$$\begin{pmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{pmatrix} = \begin{pmatrix} a_1 & a_2 & a_3 \\ b_1+3a_1 & b_2+3a_2 & b_3+3a_3 \\ c_1 & c_2 & c_3 \end{pmatrix},$$

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{pmatrix} = \begin{pmatrix} a_1 & a_2 & a_3 \\ c_1 & c_2 & c_3 \\ b_1 & b_2 & b_3 \end{pmatrix}.$$

Proposition Any elementary row operation can be simulated as left multiplication by a certain matrix.

Elementary matrices

$$E = \begin{pmatrix} 1 & & & & & & & \\ \vdots & \ddots & & & & & & \\ 0 & \cdots & 1 & & & & & \\ \vdots & & \vdots & \ddots & & & & \\ 0 & \cdots & r & \cdots & 1 & & & \\ \vdots & & \vdots & & \vdots & \ddots & & \\ 0 & \cdots & 0 & \cdots & 0 & \cdots & 1 & \end{pmatrix} \begin{array}{l} \text{row \#}i \\ \\ \text{row \#}j \end{array}$$

To obtain the matrix EA from A , add r times the i th row to the j th row. To obtain the matrix AE from A , add r times the j th column to the i th column.

Why does it work? (continued)

Assume that a square matrix A can be converted to the identity matrix by a sequence of elementary row operations. Then $E_k E_{k-1} \dots E_2 E_1 A = I$, where E_1, E_2, \dots, E_k are elementary matrices simulating those operations.

Applying the same sequence of operations to the identity matrix, we obtain the matrix

$$B = E_k E_{k-1} \dots E_2 E_1 I = E_k E_{k-1} \dots E_2 E_1.$$

Thus $BA = I$. Besides, B is invertible since elementary matrices are invertible (why?). It follows that $A = B^{-1}$, then $B = A^{-1}$.

Transpose of a matrix

Definition. Given a matrix A , the **transpose** of A , denoted A^T , is the matrix whose rows are columns of A (and whose columns are rows of A). That is, if $A = (a_{ij})$ then $A^T = (b_{ij})$, where $b_{ij} = a_{ji}$.

Examples.
$$\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix}^T = \begin{pmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{pmatrix},$$

$$\begin{pmatrix} 7 \\ 8 \\ 9 \end{pmatrix}^T = (7, 8, 9), \quad \begin{pmatrix} 4 & 7 \\ 7 & 0 \end{pmatrix}^T = \begin{pmatrix} 4 & 7 \\ 7 & 0 \end{pmatrix}.$$

Properties of transposes:

- $(A^T)^T = A$
- $(A + B)^T = A^T + B^T$
- $(rA)^T = rA^T$
- $(AB)^T = B^T A^T$
- $(A_1 A_2 \dots A_k)^T = A_k^T \dots A_2^T A_1^T$
- $(A^{-1})^T = (A^T)^{-1}$

Definition. A square matrix A is said to be **symmetric** if $A^T = A$.

For example, any diagonal matrix is symmetric.

Proposition For any square matrix A the matrices $B = AA^T$ and $C = A + A^T$ are symmetric.

Proof:

$$B^T = (AA^T)^T = (A^T)^T A^T = AA^T = B,$$

$$C^T = (A + A^T)^T = A^T + (A^T)^T = A^T + A = C.$$