#### MATH 311

Topics in Applied Mathematics I

Lecture 15:

Basis and dimension.

# **Spanning set**

Let S be a subset of a vector space V.

Definition. The **span** of the set S is the smallest subspace  $W \subset V$  that contains S. If S is not empty then  $W = \operatorname{Span}(S)$  consists of all linear combinations  $r_1\mathbf{v}_1 + r_2\mathbf{v}_2 + \cdots + r_k\mathbf{v}_k$  such that  $\mathbf{v}_1, \ldots, \mathbf{v}_k \in S$  and  $r_1, \ldots, r_k \in \mathbb{R}$ .

We say that the set S spans the subspace W or that S is a spanning set for W.

Remarks. • If  $S_1$  is a spanning set for a vector space V and  $S_1 \subset S_2 \subset V$ , then  $S_2$  is also a spanning set for V.

• If  $\mathbf{v}_0, \mathbf{v}_1, \dots, \mathbf{v}_k$  is a spanning set for V and  $\mathbf{v}_0$  is a linear combination of vectors  $\mathbf{v}_1, \dots, \mathbf{v}_k$  then  $\mathbf{v}_1, \dots, \mathbf{v}_k$  is also a spanning set for V.

# Linear independence

*Definition.* Let V be a vector space. Vectors  $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_k \in V$  are called **linearly dependent** if they satisfy a relation

$$r_1\mathbf{v}_1+r_2\mathbf{v}_2+\cdots+r_k\mathbf{v}_k=\mathbf{0},$$

where the coefficients  $r_1, \ldots, r_k \in \mathbb{R}$  are not all equal to zero. Otherwise the vectors  $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k$  are called **linearly independent**. That is, if

$$r_1\mathbf{v}_1+r_2\mathbf{v}_2+\cdots+r_k\mathbf{v}_k=\mathbf{0} \implies r_1=\cdots=r_k=0.$$

A set  $S \subset V$  is **linearly dependent** if one can find some distinct linearly dependent vectors  $\mathbf{v}_1, \dots, \mathbf{v}_k$  in S. Otherwise S is **linearly independent**.

**Theorem** Vectors  $\mathbf{v}_1, \dots, \mathbf{v}_k \in V$  are linearly dependent if and only if one of them is a linear combination of the other k-1 vectors.

### **Basis**

Definition. Let V be a vector space. Any linearly independent spanning set for V is called a **basis**.

Suppose that a set  $S \subset V$  is a basis for V.

"Spanning set" means that any vector  $\mathbf{v} \in V$  can be represented as a linear combination

$$\mathbf{v} = r_1 \mathbf{v}_1 + r_2 \mathbf{v}_2 + \cdots + r_k \mathbf{v}_k,$$

where  $\mathbf{v}_1, \dots, \mathbf{v}_k$  are distinct vectors from S and  $r_1, \dots, r_k \in \mathbb{R}$ . "Linearly independent" implies that the above representation is unique:

$$\mathbf{v} = r_1 \mathbf{v}_1 + r_2 \mathbf{v}_2 + \dots + r_k \mathbf{v}_k = r'_1 \mathbf{v}_1 + r'_2 \mathbf{v}_2 + \dots + r'_k \mathbf{v}_k$$

$$\implies (r_1 - r'_1) \mathbf{v}_1 + (r_2 - r'_2) \mathbf{v}_2 + \dots + (r_k - r'_k) \mathbf{v}_k = \mathbf{0}$$

$$\implies r_1 - r'_1 = r_2 - r'_2 = \dots = r_k - r'_k = \mathbf{0}$$

Examples. • Standard basis for  $\mathbb{R}^n$ :  $\mathbf{e}_1 = (1, 0, 0, \dots, 0, 0), \ \mathbf{e}_2 = (0, 1, 0, \dots, 0, 0), \dots$ 

$$\mathbf{e}_n = (0, 0, 0, \dots, 0, 1).$$
  
Indeed,  $(x_1, x_2, \dots, x_n) = x_1 \mathbf{e}_1 + x_2 \mathbf{e}_2 + \dots + x_n \mathbf{e}_n.$ 

• Matrices  $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ ,  $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ ,  $\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$ ,  $\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$ 

form a basis for 
$$\mathcal{M}_{2,2}(\mathbb{R})$$
. 
$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = a \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + b \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} + c \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} + d \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}.$$

• Polynomials 1 x 
$$x^2$$
  $x^{n-1}$  form a basis for

- Polynomials  $1, x, x^2, \dots, x^{n-1}$  form a basis for  $\mathcal{P}_n = \{a_0 + a_1x + \dots + a_{n-1}x^{n-1} : a_i \in \mathbb{R}\}.$
- The infinite set  $\{1, x, x^2, \dots, x^n, \dots\}$  is a basis for  $\mathcal{P}$ , the space of all polynomials.

Let  $\mathbf{v}, \mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_k \in \mathbb{R}^n$  and  $r_1, r_2, \dots, r_k \in \mathbb{R}$ .

The vector equation  $r_1\mathbf{v}_1+r_2\mathbf{v}_2+\cdots+r_k\mathbf{v}_k=\mathbf{v}$  is equivalent to the matrix equation  $A\mathbf{x}=\mathbf{v}$ , where

$$A = (\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_k), \qquad \mathbf{x} = \begin{pmatrix} r_1 \\ \vdots \\ r_k \end{pmatrix}.$$

$$r_1\begin{pmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{n1} \end{pmatrix} + r_2\begin{pmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{n2} \end{pmatrix} + \cdots + r_k\begin{pmatrix} a_{1k} \\ a_{2k} \\ \vdots \\ a_{nk} \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix} \iff$$

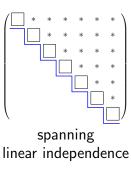
$$\begin{pmatrix} a_{11} & a_{12} & \dots & a_{1k} \\ a_{21} & a_{22} & \dots & a_{2k} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nk} \end{pmatrix} \begin{pmatrix} r_1 \\ r_2 \\ \vdots \\ r_k \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix} \iff A\mathbf{x} = \mathbf{v}$$

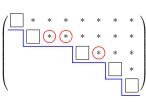
Let  $\mathbf{v}, \mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_k \in \mathbb{R}^n$  and  $r_1, r_2, \dots, r_k \in \mathbb{R}$ . The vector equation  $r_1\mathbf{v}_1 + r_2\mathbf{v}_2 + \dots + r_k\mathbf{v}_k = \mathbf{v}$  is equivalent to the matrix equation  $A\mathbf{x} = \mathbf{v}$ , where

$$A = (\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_k), \qquad \mathbf{x} = \begin{pmatrix} r_1 \\ \vdots \\ r_k \end{pmatrix}.$$

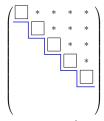
That is, A is the  $n \times k$  matrix such that vectors  $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_k$  are consecutive columns of A.

- Vectors  $\mathbf{v}_1, \dots, \mathbf{v}_k$  span  $\mathbb{R}^n$  if the row echelon form of A has no zero rows.
- Vectors  $\mathbf{v}_1, \dots, \mathbf{v}_k$  are linearly independent if the row echelon form of A has a leading entry in each column (no free variables).

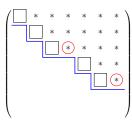




spanning no linear independence



no spanning linear independence



no spanning no linear independence

## Bases for $\mathbb{R}^n$

Let  $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_k$  be vectors in  $\mathbb{R}^n$ .

**Theorem 1** If k < n then the vectors  $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_k$  do not span  $\mathbb{R}^n$ .

**Theorem 2** If k > n then the vectors  $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_k$  are linearly dependent.

**Theorem 3** If k = n then the following conditions are equivalent:

- (i)  $\{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n\}$  is a basis for  $\mathbb{R}^n$ ;
- (ii)  $\{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n\}$  is a spanning set for  $\mathbb{R}^n$ ;
- (iii)  $\{v_1, v_2, \dots, v_n\}$  is a linearly independent set.

Example. Consider vectors  $\mathbf{v}_1 = (1, -1, 1)$ ,  $\mathbf{v}_2 = (1, 0, 0)$ ,  $\mathbf{v}_3 = (1, 1, 1)$ , and  $\mathbf{v}_4 = (1, 2, 4)$  in  $\mathbb{R}^3$ .

Vectors  $\mathbf{v}_1$  and  $\mathbf{v}_2$  are linearly independent (as they are not parallel), but they do not span  $\mathbb{R}^3$ .

Vectors  $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$  are linearly independent since

$$\begin{vmatrix} 1 & 1 & 1 \\ -1 & 0 & 1 \\ 1 & 0 & 1 \end{vmatrix} = - \begin{vmatrix} -1 & 1 \\ 1 & 1 \end{vmatrix} = -(-2) = 2 \neq 0.$$

Therefore  $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$  is a basis for  $\mathbb{R}^3$ .

Vectors  $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4$  span  $\mathbb{R}^3$  (because  $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$  already span  $\mathbb{R}^3$ ), but they are linearly dependent.

#### **Dimension**

**Theorem 1** Any vector space has a basis.

**Theorem 2** If a vector space V has a finite basis, then all bases for V are finite and have the same number of elements.

Definition. The **dimension** of a vector space V, denoted dim V, is the number of elements in any of its bases.

Examples. • dim  $\mathbb{R}^n = n$ 

•  $\mathcal{M}_{2,2}(\mathbb{R})$ : the space of 2×2 matrices dim  $\mathcal{M}_{2,2}(\mathbb{R})=4$ 

•  $\mathcal{M}_{m,n}(\mathbb{R})$ : the space of  $m \times n$  matrices  $\dim \mathcal{M}_{m,n}(\mathbb{R}) = mn$ 

- $\mathcal{P}_n$ : polynomials of degree less than n dim  $\mathcal{P}_n = n$
- $\bullet$   $\ensuremath{\mathcal{P}}$  : the space of all polynomials  $\dim \ensuremath{\mathcal{P}} = \infty$
- $\bullet \ \ \{ {\bm 0} \} \text{: the trivial vector space} \\ \dim \left\{ {\bm 0} \right\} = 0$

**Problem.** Find the dimension of the plane x + 2z = 0 in  $\mathbb{R}^3$ .

The general solution of the equation x+2z=0 is  $\begin{cases} x=-2s \\ y=t \\ z=s \end{cases}$   $(t,s\in\mathbb{R})$ 

That is, (x, y, z) = (-2s, t, s) = t(0, 1, 0) + s(-2, 0, 1). Hence the plane is the span of vectors  $\mathbf{v}_1 = (0, 1, 0)$  and  $\mathbf{v}_2 = (-2, 0, 1)$ . These vectors are linearly independent as they are not parallel.

Thus  $\{\mathbf{v}_1, \mathbf{v}_2\}$  is a basis so that the dimension of the plane is 2.

### How to find a basis?

**Theorem** Let S be a subset of a vector space V. Then the following conditions are equivalent:

- (i) S is a linearly independent spanning set for V, i.e., a basis;
- (ii) S is a minimal spanning set for V;
- (iii) S is a maximal linearly independent subset of V.

"Minimal spanning set" means "remove any element from this set, and it is no longer a spanning set".

"Maximal linearly independent subset" means "add any element of V to this set, and it will become linearly dependent".

**Theorem** Let V be a vector space. Then **(i)** any spanning set for V can be reduced to a minimal spanning set;

(ii) any linearly independent subset of V can be extended to a maximal linearly independent set.

**Corollary 1** Any spanning set contains a basis while any linearly independent set is contained in a basis.

**Corollary 2** A vector space is finite-dimensional if and only if it is spanned by a finite set.