

MATH 311

Topics in Applied Mathematics I

Lecture 32:

Area and volume.

Multiple integrals.

Area, volume, and determinants

- 2×2 determinants and plane geometry

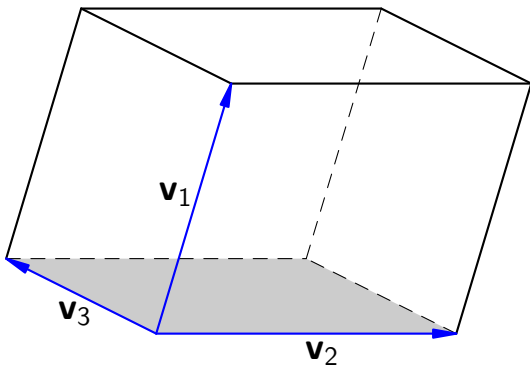
Let P be a parallelogram in the plane \mathbb{R}^2 . Suppose that vectors $\mathbf{v}_1, \mathbf{v}_2 \in \mathbb{R}^2$ are represented by adjacent sides of P . Then $\text{area}(P) = |\det A|$, where $A = (\mathbf{v}_1, \mathbf{v}_2)$, a matrix whose columns are \mathbf{v}_1 and \mathbf{v}_2 .

Consider a linear operator $L_A : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ given by $L_A(\mathbf{v}) = A\mathbf{v}$ for any column vector \mathbf{v} . Then $\text{area}(L_A(D)) = |\det A| \text{area}(D)$ for any bounded domain D .

- 3×3 determinants and space geometry

Let Π be a parallelepiped in space \mathbb{R}^3 . Suppose that vectors $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3 \in \mathbb{R}^3$ are represented by adjacent edges of Π . Then $\text{volume}(\Pi) = |\det B|$, where $B = (\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3)$, a matrix whose columns are $\mathbf{v}_1, \mathbf{v}_2$, and \mathbf{v}_3 .

Similarly, $\text{volume}(L_B(D)) = |\det B| \text{volume}(D)$ for any bounded domain $D \subset \mathbb{R}^3$.



$\text{volume}(\Pi) = |\det B|$, where $B = (\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3)$. Note that the parallelepiped Π is the image under L_B of a unit cube whose adjacent edges are $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$.

The triple $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ obeys the right-hand rule. We say that L_B **preserves orientation** if it preserves the hand rule for any basis. This is the case if and only if $\det B > 0$.

Riemann sums in two dimensions

Consider a closed coordinate rectangle

$$R = [a, b] \times [c, d] \subset \mathbb{R}^2.$$

Definition. A **Riemann sum** of a function $f : R \rightarrow \mathbb{R}$ with respect to a partition $P = \{D_1, D_2, \dots, D_n\}$ of R generated by samples $t_j \in D_j$ is a sum

$$\mathcal{S}(f, P, t_j) = \sum_{j=1}^n f(t_j) \text{area}(D_j).$$

The norm of the partition P is $\|P\| = \max_{1 \leq j \leq n} \text{diam}(D_j)$.

Definition. The Riemann sums $\mathcal{S}(f, P, t_j)$ **converge** to a limit $I(f)$ as the norm $\|P\| \rightarrow 0$ if for every $\varepsilon > 0$ there exists $\delta > 0$ such that $\|P\| < \delta$ implies $|\mathcal{S}(f, P, t_j) - I(f)| < \varepsilon$ for any partition P and choice of samples t_j .

If this is the case, then the function f is called **integrable** on R and the limit $I(f)$ is called the **integral** of f over R .

Double integral

Closed coordinate rectangle $R = [a, b] \times [c, d]$
 $= \{(x, y) \in \mathbb{R}^2 \mid a \leq x \leq b, c \leq y \leq d\}$.

Notation: $\iint_R f \, dA$ or $\iint_R f(x, y) \, dx \, dy$.

Theorem 1 If f is continuous on the closed rectangle R , then f is integrable.

Theorem 2 A function $f : R \rightarrow \mathbb{R}$ is Riemann integrable on the rectangle R if and only if f is bounded on R and continuous almost everywhere on R (that is, the set of discontinuities of f has zero area).

Fubini's Theorem

Fubini's Theorem allows us to reduce a multiple integral to a repeated one-dimensional integral.

Theorem If a function f is integrable on $R = [a, b] \times [c, d]$, then

$$\begin{aligned}\iint_R f \, dA &= \int_a^b \left(\int_c^d f(x, y) \, dy \right) dx \\ &= \int_c^d \left(\int_a^b f(x, y) \, dx \right) dy.\end{aligned}$$

In particular, this implies that we can change the order of integration in a repeated integral.

Integrals over general domains

Suppose $f : D \rightarrow \mathbb{R}$ is a function defined on a (Jordan) measurable set $D \subset \mathbb{R}^2$. Since D is bounded, it is contained in a rectangle R . To define the integral of f over D , we extend the function f to a function on R :

$$f^{\text{ext}}(x, y) = \begin{cases} f(x, y) & \text{if } (x, y) \in D, \\ 0 & \text{if } (x, y) \notin D. \end{cases}$$

Definition. $\iint_D f \, dA$ is defined to be $\iint_R f^{\text{ext}} \, dA$.

In particular, $\text{area}(D) = \iint_D 1 \, dA$.

Integration as a linear operation

Theorem 1 If functions f, g are integrable on a set $D \subset \mathbb{R}^2$, then the sum $f + g$ is also integrable on D and

$$\iint_D (f + g) dA = \iint_D f dA + \iint_D g dA.$$

Theorem 2 If a function f is integrable on a set $D \subset \mathbb{R}^2$, then for each $\alpha \in \mathbb{R}$ the scalar multiple αf is also integrable on D and

$$\iint_D \alpha f dA = \alpha \iint_D f dA.$$

More properties of integrals

Theorem 3 If functions f, g are integrable on a set $D \subset \mathbb{R}^2$, and $f(x, y) \leq g(x, y)$ for all $(x, y) \in D$, then

$$\iint_D f \, dA \leq \iint_D g \, dA.$$

Theorem 4 If a function f is integrable on sets $D_1, D_2 \subset \mathbb{R}^2$, then it is integrable on their union $D_1 \cup D_2$. Moreover, if the sets D_1 and D_2 are disjoint up to a set of zero area, then

$$\iint_{D_1 \cup D_2} f \, dA = \iint_{D_1} f \, dA + \iint_{D_2} f \, dA.$$

Change of variables in a double integral

Theorem Let $D \subset \mathbb{R}^2$ be a measurable domain and f be an integrable function on D . If $\mathbf{T} = (u, v)$ is a smooth coordinate mapping such that \mathbf{T}^{-1} is defined on D , then

$$\begin{aligned} & \iint_D f(u, v) \, du \, dv \\ &= \iint_{\mathbf{T}^{-1}(D)} f(u(x, y), v(x, y)) \left| \det \frac{\partial(u, v)}{\partial(x, y)} \right| \, dx \, dy. \end{aligned}$$

In particular, the integral in the right-hand side is well defined.

Problem Evaluate a double integral

$$\iint_P (2x + 3y - \cos(\pi x + 2\pi y)) \, dx \, dy$$

over a parallelogram P with vertices $(-1, -1)$, $(1, 0)$, $(2, 2)$, and $(0, 1)$.

Adjacent edges of the parallelogram P are represented by vectors $\mathbf{v}_1 = (1, 0) - (-1, -1) = (2, 1)$ and $\mathbf{v}_2 = (0, 1) - (-1, -1) = (1, 2)$.

Consider a transformation L of the plane \mathbb{R}^2 given by

$$L \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} + \begin{pmatrix} -1 \\ -1 \end{pmatrix} = \begin{pmatrix} 2u + v - 1 \\ u + 2v - 1 \end{pmatrix}$$

(columns of the matrix are vectors \mathbf{v}_1 and \mathbf{v}_2). By construction, L maps the unit square $[0, 1] \times [0, 1]$ onto the parallelogram P . The Jacobian matrix J of L is the same at

any point: $J = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$.

Changing coordinates in the integral from (x, y) to (u, v) so that $(x, y) = L(u, v) = (2u + v - 1, u + 2v - 1)$, we obtain

$$\begin{aligned} & \iint_P (2x + 3y - \cos(\pi x + 2\pi y)) \, dx \, dy \\ &= \iint_{L^{-1}(P)} (7u + 8v - 5 - \cos(4\pi u + 5\pi v - 3\pi)) |\det J| \, du \, dv \\ &= \int_0^1 \int_0^1 3(7u + 8v - 5 + \cos(4\pi u + 5\pi v)) \, du \, dv \\ &= \frac{21}{2} + 12 - 15 + \int_0^1 \int_0^1 3 \cos(4\pi u + 5\pi v) \, du \, dv. \end{aligned}$$

$$\begin{aligned} \text{Further, } & \int_0^1 3 \cos(4\pi u + 5\pi v) \, du = \frac{3}{4\pi} \sin(4\pi u + 5\pi v) \Big|_{u=0}^1 \\ &= \frac{3}{4\pi} (\sin(4\pi + 5\pi v) - \sin(5\pi v)) = 0 \text{ for all } v. \end{aligned}$$

$$\text{It follows that } \iint_P (2x + 3y - \cos(\pi x + 2\pi y)) \, dx \, dy = \frac{15}{2}.$$

Triple integral

To integrate in \mathbb{R}^3 , volumes are used instead of areas in \mathbb{R}^2 . Instead of coordinate rectangles, basic sets are coordinate boxes (or bricks) $B = [a_1, b_1] \times [a_2, b_2] \times [a_3, b_3] \subset \mathbb{R}^3$. Then we can define an integral of a function f over a measurable set $D \subset \mathbb{R}^3$.

Notation: $\iiint_D f \, dV$ or $\iiint_D f(x, y, z) \, dx \, dy \, dz$.

The properties of triple integrals are completely analogous to those of double integrals. In particular, Fubini's Theorem is formulated as follows.

Theorem If a function f is integrable on a brick $B = [a_1, b_1] \times [a_2, b_2] \times [a_3, b_3] \subset \mathbb{R}^3$, then

$$\iiint_B f \, dV = \int_{a_1}^{b_1} \left(\int_{a_2}^{b_2} \left(\int_{a_3}^{b_3} f(x, y, z) \, dz \right) dy \right) dx.$$