MATH 323 Linear Algebra

Lecture 6: Matrix algebra (continued). Determinants.

Elementary matrices

Theorem 1 Any elementary row operation σ on matrices with *n* rows can be simulated as left multiplication by a certain $n \times n$ matrix E_{σ} (called an *elementary matrix*).

Theorem 2 Elementary matrices are invertible.

Proof: Suppose E_{σ} is an $n \times n$ elementary matrix corresponding to an operation σ . We know that σ can be undone by another elementary row operation τ . It is easy to check that σ undoes τ as well. Then for any matrix A with nrows we have $E_{\tau}E_{\sigma}A = A$ (since τ undoes σ) and $E_{\sigma}E_{\tau}A = A$ (since σ undoes τ). In particular, $E_{\tau}E_{\sigma}I = E_{\sigma}E_{\tau}I = I$, which implies that $E_{\tau} = E_{\sigma}^{-1}$.

Theorem 3 A square matrix is invertible if and only if it can be expanded into a product of elementary matrices.

Theorem Suppose that a sequence of elementary row operations converts a matrix *A* into the identity matrix.

Then the same sequence of operations converts the identity matrix into the inverse matrix A^{-1} .

Proof: Let E_1, E_2, \ldots, E_k be elementary matrices that correspond to elementary row operations converting A into I. Then $E_k E_{k-1} \ldots E_2 E_1 A = I$.

Applying the same sequence of operations to the identity matrix I, we obtain the matrix

$$B = E_k E_{k-1} \dots E_2 E_1 I = E_k E_{k-1} \dots E_2 E_1.$$

Therefore BA = I. Besides, B is invertible since elementary matrices are invertible. Then $B^{-1}(BA) = B^{-1}I$. It follows that $A = B^{-1}$, hence $B = A^{-1}$.

Theorem A square matrix A is invertible if and only if $\mathbf{x} = \mathbf{0}$ is the only solution of the matrix equation $A\mathbf{x} = \mathbf{0}$.

Corollary 1 For any $n \times n$ matrices A and B,

$$BA = I \iff AB = I.$$

Proof: It is enough to prove that $BA = I \implies AB = I$. Assume BA = I. Then $A\mathbf{x} = \mathbf{0} \implies B(A\mathbf{x}) = B\mathbf{0}$ $\implies (BA)\mathbf{x} = \mathbf{0} \implies \mathbf{x} = \mathbf{0}$. By the theorem, A is invertible. Then $BA = I \implies A(BA)A^{-1} = AIA^{-1} \implies AB = I$.

Corollary 2 Suppose *A* and *B* are $n \times n$ matrices. If the product *AB* is invertible, then both *A* and *B* are invertible.

Proof: Let
$$C = B(AB)^{-1}$$
 and $D = (AB)^{-1}A$. Then
 $AC = A(B(AB)^{-1}) = (AB)(AB)^{-1} = I$ and
 $DB = ((AB)^{-1}A)B = (AB)^{-1}(AB) = I$. By Corollary 1,
 $C = A^{-1}$ and $D = B^{-1}$.

Transpose of a matrix

Definition. Given a matrix A, the **transpose** of A, denoted A^{T} , is the matrix whose rows are columns of A (and whose columns are rows of A). That is, if $A = (a_{ij})$ then $A^{T} = (b_{ij})$, where $b_{ij} = a_{ji}$.

Examples.
$$\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix}^{T} = \begin{pmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{pmatrix}$$
,
 $\begin{pmatrix} 7 \\ 8 \\ 9 \end{pmatrix}^{T} = (7, 8, 9), \qquad \begin{pmatrix} 4 & 7 \\ 7 & 0 \end{pmatrix}^{T} = \begin{pmatrix} 4 & 7 \\ 7 & 0 \end{pmatrix}$

Properties of transposes:

•
$$(A^T)^T = A$$

• $(A+B)^T = A^T + B^T$

•
$$(rA)^T = rA^T$$

- $(AB)^T = B^T A^T$
- $(A_1A_2\ldots A_k)^T = A_k^T\ldots A_2^TA_1^T$

•
$$(A^{-1})^T = (A^T)^{-1}$$

Definition. A square matrix A is said to be symmetric if $A^T = A$.

For example, any diagonal matrix is symmetric.

Proposition For any square matrix A the matrices $B = AA^T$ and $C = A + A^T$ are symmetric.

Proof:

$$B^{T} = (AA^{T})^{T} = (A^{T})^{T}A^{T} = AA^{T} = B,$$

 $C^{T} = (A + A^{T})^{T} = A^{T} + (A^{T})^{T} = A^{T} + A = C.$

Determinants

Determinant is a scalar assigned to each square matrix.

Notation. The determinant of a matrix $A = (a_{ii})_{1 \le i, i \le n}$ is denoted det A or

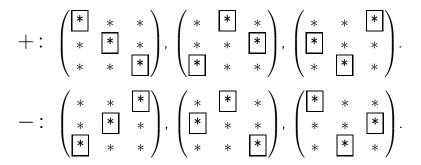
I					
	a_{11}	a_{12}	• • •	a _{1n}	
	a ₂₁	<i>a</i> ₂₂	• • •	a _{2n}	
	÷	÷	•••	÷	•
	a _{n1}	a _{n2}	•••	a _{nn}	

Principal property: det $A \neq 0$ if and only if a system of linear equations with the coefficient matrix A has a unique solution. Equivalently, det $A \neq 0$ if and only if the matrix A is invertible.

Definition in low dimensions

Definition. det (a) = a,
$$\begin{vmatrix} a & b \\ c & d \end{vmatrix}$$
 = ad - bc,

 $\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} - a_{13}a_{22}a_{31} - a_{12}a_{21}a_{33} - a_{11}a_{23}a_{32}.$



Examples: 2×2 matrices

$$\begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = 1, \qquad \begin{vmatrix} 3 & 0 \\ 0 & -4 \end{vmatrix} = -12,$$
$$\begin{vmatrix} -2 & 5 \\ 0 & 3 \end{vmatrix} = -6, \qquad \begin{vmatrix} 7 & 0 \\ 5 & 2 \end{vmatrix} = 14,$$
$$\begin{vmatrix} 0 & -1 \\ 1 & 0 \end{vmatrix} = 1, \qquad \begin{vmatrix} 0 & 0 \\ 4 & 1 \end{vmatrix} = 0,$$
$$\begin{vmatrix} -1 & 3 \\ -1 & 3 \end{vmatrix} = 0, \qquad \begin{vmatrix} 2 & 1 \\ 8 & 4 \end{vmatrix} = 0.$$

Examples: 3×3 matrices

$$\begin{vmatrix} 3 & -2 & 0 \\ 1 & 0 & 1 \\ -2 & 3 & 0 \end{vmatrix} = 3 \cdot 0 \cdot 0 + (-2) \cdot 1 \cdot (-2) + 0 \cdot 1 \cdot 3 - \\ -0 \cdot 0 \cdot (-2) - (-2) \cdot 1 \cdot 0 - 3 \cdot 1 \cdot 3 = 4 - 9 = -5, \\\begin{vmatrix} 1 & 4 & 6 \\ 0 & 2 & 5 \\ 0 & 0 & 3 \end{vmatrix} = 1 \cdot 2 \cdot 3 + 4 \cdot 5 \cdot 0 + 6 \cdot 0 \cdot 0 - \end{vmatrix}$$

 $-6 \cdot 2 \cdot 0 - 4 \cdot 0 \cdot 3 - 1 \cdot 5 \cdot 0 = 1 \cdot 2 \cdot 3 = 6.$

General definition

The general definition of the determinant is quite complicated as there is no simple explicit formula.

There are several approaches to defining determinants.

Approach 1 (original): an explicit (but very complicated) formula.

Approach 2 (axiomatic): we formulate properties that the determinant should have.

Approach 3 (inductive): the determinant of an $n \times n$ matrix is defined in terms of determinants of certain $(n-1) \times (n-1)$ matrices.

Classical definition

Definition. If $A = (a_{ij})$ is an $n \times n$ matrix then $\det A = \sum_{\pi \in S_n} \operatorname{sgn}(\pi) a_{1,\pi(1)} a_{2,\pi(2)} \dots a_{n,\pi(n)},$

where π runs over S_n , the set of all permutations of $\{1, 2, ..., n\}$, and $sgn(\pi)$ denotes the sign of the permutation π .

Remarks. • A **permutation** of the set $\{1, 2, ..., n\}$ is an invertible mapping of this set onto itself. There are n! such mappings.

• The sign $sgn(\pi)$ can be 1 or -1. Its definition is rather complicated.

Axiomatic definition

 $\mathcal{M}_{n,n}(\mathbb{R})$: the set of $n \times n$ matrices with real entries.

Theorem There exists a unique function det : $\mathcal{M}_{n,n}(\mathbb{R}) \to \mathbb{R}$ (called the determinant) with the following properties:

(D1) if a row of a matrix is multiplied by a scalar *r*, the determinant is also multiplied by *r*;

(D2) if we add a row of a matrix multiplied by a scalar to another row, the determinant remains the same;

(D3) if we interchange two rows of a matrix, the determinant changes its sign;

(D4) det I = 1.

Corollary 1 Suppose A is a square matrix and B is obtained from A applying elementary row operations. Then det A = 0 if and only if det B = 0.

Corollary 2 det B = 0 whenever the matrix B has a zero row.

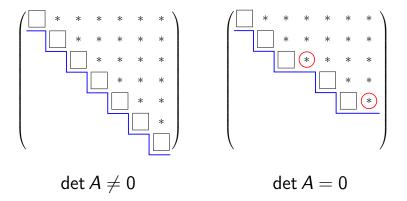
Hint: Multiply the zero row by the zero scalar.

Corollary 3 det A = 0 if and only if the matrix A is not invertible.

Idea of the proof: Let *B* be the reduced row echelon form of *A*. If *A* is invertible then B = I; otherwise *B* has a zero row.

Remark. The same argument proves that properties (D1)-(D4) are enough to evaluate any determinant.

Row echelon form of a square matrix A:



Example.
$$A = \begin{pmatrix} 3 & -2 & 0 \\ 1 & 0 & 1 \\ -2 & 3 & 0 \end{pmatrix}$$
, det $A = ?$

In the previous lecture, we have transformed the matrix A into the identity matrix using elementary row operations:

- interchange the 1st row with the 2nd row,
- add -3 times the 1st row to the 2nd row,
- add 2 times the 1st row to the 3rd row,
- multiply the 2nd row by -0.5,
- add -3 times the 2nd row to the 3rd row,
- multiply the 3rd row by -0.4,
- add -1.5 times the 3rd row to the 2nd row,
- add -1 times the 3rd row to the 1st row.

Example.
$$A = \begin{pmatrix} 3 & -2 & 0 \\ 1 & 0 & 1 \\ -2 & 3 & 0 \end{pmatrix}$$
, det $A = ?$

In the previous lecture, we have transformed the matrix A into the identity matrix using elementary row operations.

These included two row multiplications, by -0.5 and by -0.4, and one row exchange.

It follows that

F

det
$$I = -(-0.5)(-0.4)$$
 det $A = (-0.2)$ det A .
lence det $A = -5$ det $I = -5$.