MATH 323
Linear Algebra

Lecture 9:
Subspaces of vector spaces.
Span. Spanning set.

Abstract vector space

A *vector space* is a set V equipped with two operations, **addition** $V \times V \ni (\mathbf{x}, \mathbf{y}) \mapsto \mathbf{x} + \mathbf{y} \in V$ and **scalar multiplication** $\mathbb{R} \times V \ni (r, \mathbf{x}) \mapsto r\mathbf{x} \in V$, that have the following properties:

A1. $\mathbf{x} + \mathbf{y} = \mathbf{y} + \mathbf{x}$ for all $\mathbf{x}, \mathbf{y} \in V$;

A2. $(\mathbf{x} + \mathbf{y}) + \mathbf{z} = \mathbf{x} + (\mathbf{y} + \mathbf{z})$ for all $\mathbf{x}, \mathbf{y}, \mathbf{z} \in V$;

A3. there exists an element of V, called the *zero vector* and denoted $\mathbf{0}$, such that $\mathbf{x} + \mathbf{0} = \mathbf{0} + \mathbf{x} = \mathbf{x}$ for all $\mathbf{x} \in V$;

A4. for any $\mathbf{x} \in V$ there exists an element of V, denoted $-\mathbf{x}$, such that $\mathbf{x} + (-\mathbf{x}) = (-\mathbf{x}) + \mathbf{x} = \mathbf{0}$;

A5. $r(\mathbf{x} + \mathbf{y}) = r\mathbf{x} + r\mathbf{y}$ for all $r \in \mathbb{R}$ and $\mathbf{x}, \mathbf{y} \in V$;

A6. $(r + s)\mathbf{x} = r\mathbf{x} + s\mathbf{x}$ for all $r, s \in \mathbb{R}$ and $\mathbf{x} \in V$;

A7. $(rs)\mathbf{x} = r(s\mathbf{x})$ for all $r, s \in \mathbb{R}$ and $\mathbf{x} \in V$;

A8. $1\mathbf{x} = \mathbf{x}$ for all $\mathbf{x} \in V$.

Additional properties of vector spaces

- The zero vector is unique.
- For any $\mathbf{x} \in V$, the negative $-\mathbf{x}$ is unique.
- $\mathbf{x} + \mathbf{y} = \mathbf{z} \iff \mathbf{x} = \mathbf{z} - \mathbf{y}$ for all $\mathbf{x}, \mathbf{y}, \mathbf{z} \in V$.
- $\mathbf{x} + \mathbf{z} = \mathbf{y} + \mathbf{z} \iff \mathbf{x} = \mathbf{y}$ for all $\mathbf{x}, \mathbf{y}, \mathbf{z} \in V$.
- $0\mathbf{x} = \mathbf{0}$ for any $\mathbf{x} \in V$.
- $(-1)\mathbf{x} = -\mathbf{x}$ for any $\mathbf{x} \in V$.

Examples of vector spaces

- $\mathbb{R}^n$: n-dimensional coordinate vectors
- $\mathcal{M}_{m,n}(\mathbb{R})$: $m \times n$ matrices with real entries
- $\mathbb{R}^\infty$: infinite sequences $(x_1, x_2, \dots)$, $x_i \in \mathbb{R}$
- $\{\mathbf{0}\}$: the trivial vector space
- $F(\mathbb{R})$: the set of all functions $f : \mathbb{R} \to \mathbb{R}$
- $C(\mathbb{R})$: all continuous functions $f : \mathbb{R} \to \mathbb{R}$
- $C^1(\mathbb{R})$: all continuously differentiable functions $f : \mathbb{R} \to \mathbb{R}$
- $C^\infty(\mathbb{R})$: all smooth functions $f : \mathbb{R} \to \mathbb{R}$
- $\mathcal{P}$: all polynomials $p(x) = a_0 + a_1 x + \cdots + a_n x^n$

Subspaces of vector spaces

Definition. A vector space V_0 is a **subspace** of a vector space V if $V_0 \subset V$ and the linear operations on V_0 agree with the linear operations on V.

Examples.

- $\mathcal{F}(\mathbb{R})$: all functions $f : \mathbb{R} \to \mathbb{R}$
- $C(\mathbb{R})$: all continuous functions $f : \mathbb{R} \to \mathbb{R}$

$C(\mathbb{R})$ is a subspace of $\mathcal{F}(\mathbb{R})$.

- $\mathcal{P}$: polynomials $p(x) = a_0 + a_1 x + \cdots + a_k x^k$
- $\mathcal{P}_n$: polynomials of degree less than n

$\mathcal{P}_n$ is a subspace of $\mathcal{P}$.

Subspaces of vector spaces

Counterexamples.

- $\mathbb{R}^n$: *n*-dimensional coordinate vectors
- $\mathbb{Q}^n$: vectors with rational coordinates

$\mathbb{Q}^n$ is not a subspace of $\mathbb{R}^n$.

$\sqrt{2}(1, 1, \ldots, 1) \notin \mathbb{Q}^n \implies \mathbb{Q}^n$ is not a vector space (scaling is not well defined).

- $\mathbb{R}$ with the standard linear operations
- $\mathbb{R}_+$ with the operations $\oplus$ and $\odot$

$\mathbb{R}_+$ is not a subspace of $\mathbb{R}$ since the linear operations do not agree.

If S is a subset of a vector space V then S inherits from V addition and scalar multiplication. However S need not be closed under these operations.

Proposition A subset S of a vector space V is a subspace of V if and only if S is **nonempty** and **closed under linear operations**, i.e.,

$$\mathbf{x}, \mathbf{y} \in S \implies \mathbf{x} + \mathbf{y} \in S,$$

$$\mathbf{x} \in S \implies r\mathbf{x} \in S \text{ for all } r \in \mathbb{R}.$$

Sketch of the proof: "only if" is obvious.
"if": properties like associative, commutative, or distributive law hold for S because they hold for V. We only need to verify properties A3 and A4. Take any $\mathbf{x} \in S$ (note that S is nonempty). Then $\mathbf{0} = 0\mathbf{x} \in S$. Also, $-\mathbf{x} = (-1)\mathbf{x} \in S$. Thus $\mathbf{0}$ and $-\mathbf{x}$ in S are the same as in V.

Example. $V = \mathbb{R}^2$.

- The line $x - y = 0$ is a subspace of $\mathbb{R}^2$.

The line consists of all vectors of the form (t, t), $t \in \mathbb{R}$.

$(t, t) + (s, s) = (t + s, t + s) \implies$ closed under addition
$r(t, t) = (rt, rt) \implies$ closed under scaling

- The parabola $y = x^2$ is not a subspace of $\mathbb{R}^2$.

It is enough to find one explicit counterexample.

Counterexample 1: $(1, 1) + (-1, 1) = (0, 2)$.

$(1, 1)$ and $(-1, 1)$ lie on the parabola while $(0, 2)$ does not
$\implies$ not closed under addition

Counterexample 2: $2(1, 1) = (2, 2)$.

$(1, 1)$ lies on the parabola while $(2, 2)$ does not
$\implies$ not closed under scaling

Example. $V = \mathbb{R}^3$.

- The plane $z = 0$ is a subspace of $\mathbb{R}^3$.
- The plane $z = 1$ is not a subspace of $\mathbb{R}^3$.
- The line $t(1, 1, 0)$, $t \in \mathbb{R}$ is a subspace of $\mathbb{R}^3$ and a subspace of the plane $z = 0$.
- The line $(1, 1, 1) + t(1, -1, 0)$, $t \in \mathbb{R}$ is not a subspace of $\mathbb{R}^3$ as it lies in the plane $x + y + z = 3$, which does not contain $\mathbf{0}$.
- In general, a straight line or a plane in $\mathbb{R}^3$ is a subspace if and only if it passes through the origin.

System of linear equations:

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2 \\ \cdots\cdots\cdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n = b_m \end{cases}$$

Any solution $(x_1, x_2, \ldots, x_n)$ is an element of $\mathbb{R}^n$.

Theorem The solution set of the system is a subspace of $\mathbb{R}^n$ if and only if all $b_i = 0$.

Theorem The solution set of a system of linear equations in n variables is a subspace of $\mathbb{R}^n$ if and only if all equations are homogeneous.

Proof: "only if": the zero vector $\mathbf{0} = (0, 0, \ldots, 0)$, which belongs to every subspace, is a solution only if all equations are homogeneous.

"if": a system of homogeneous linear equations is equivalent to a matrix equation $A\mathbf{x} = \mathbf{0}$, where A is the coefficient matrix of the system and all vectors are regarded as column vectors.

$A\mathbf{0} = \mathbf{0} \implies \mathbf{0}$ is a solution $\implies$ solution set is not empty.

If $A\mathbf{x} = \mathbf{0}$ and $A\mathbf{y} = \mathbf{0}$ then $A(\mathbf{x} + \mathbf{y}) = A\mathbf{x} + A\mathbf{y} = \mathbf{0}$
$\implies$ solution set is closed under addition.

If $A\mathbf{x} = \mathbf{0}$ then $A(r\mathbf{x}) = r(A\mathbf{x}) = \mathbf{0}$
$\implies$ solution set is closed under scaling.

Thus the solution set is a subspace of $\mathbb{R}^n$.

Examples of subspaces of $\mathcal{M}_{2,2}(\mathbb{R})$*:* $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$

- diagonal matrices: $b = c = 0$
- upper triangular matrices: $c = 0$
- lower triangular matrices: $b = 0$
- symmetric matrices ($A^T = A$): $b = c$
- anti-symmetric (or skew-symmetric) matrices ($A^T = -A$): $a = d = 0,\ c = -b$
- matrices with zero trace: $a + d = 0$ (trace = the sum of diagonal entries)
- matrices with zero determinant, $ad - bc = 0$, **do not** form a subspace: $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$.

Let V be a vector space and $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n \in V$. Consider the set L of all linear combinations $r_1\mathbf{v}_1 + r_2\mathbf{v}_2 + \cdots + r_n\mathbf{v}_n$, where $r_1, r_2, \dots, r_n \in \mathbb{R}$.

Theorem L is a subspace of V.

Proof: First of all, L is not empty. For example, $\mathbf{0} = 0\mathbf{v}_1 + 0\mathbf{v}_2 + \cdots + 0\mathbf{v}_n$ belongs to L.

The set L is closed under addition since

$$(r_1\mathbf{v}_1+r_2\mathbf{v}_2+\cdots+r_n\mathbf{v}_n) + (s_1\mathbf{v}_1+s_2\mathbf{v}_2+\cdots+s_n\mathbf{v}_n) =$$
$$= (r_1+s_1)\mathbf{v}_1 + (r_2+s_2)\mathbf{v}_2 + \cdots + (r_n+s_n)\mathbf{v}_n.$$

The set L is closed under scalar multiplication since

$$t(r_1\mathbf{v}_1+r_2\mathbf{v}_2+\cdots+r_n\mathbf{v}_n) = (tr_1)\mathbf{v}_1+(tr_2)\mathbf{v}_2+\cdots+(tr_n)\mathbf{v}_n.$$

Thus L is a subspace of V.

Span: implicit definition

Let S be a subset of a vector space V.

Definition. The **span** of the set S, denoted $\mathrm{Span}(S)$, is the smallest subspace of V that contains S. That is,

- $\mathrm{Span}(S)$ is a subspace of V;
- for any subspace $W \subset V$ one has
$$S \subset W \implies \mathrm{Span}(S) \subset W.$$

Remark. The span of any set $S \subset V$ is well defined (namely, it is the intersection of all subspaces of V that contain S).

Span: effective description

Let S be a subset of a vector space V.

- If $S = \{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n\}$ then $\mathrm{Span}(S)$ is the set of all linear combinations $r_1\mathbf{v}_1 + r_2\mathbf{v}_2 + \cdots + r_n\mathbf{v}_n$, where $r_1, r_2, \dots, r_n \in \mathbb{R}$.

- If S is an infinite set then $\mathrm{Span}(S)$ is the set of all linear combinations $r_1\mathbf{u}_1 + r_2\mathbf{u}_2 + \cdots + r_k\mathbf{u}_k$, where $\mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_k \in S$ and $r_1, r_2, \dots, r_k \in \mathbb{R}$ $(k \geq 1)$.

- If S is the empty set then $\mathrm{Span}(S) = \{\mathbf{0}\}$.

Examples of subspaces of $\mathcal{M}_{2,2}(\mathbb{R})$:

- The span of $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ and $\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$ consists of all matrices of the form

$$a\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + b\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}.$$

This is the subspace of diagonal matrices.

- The span of $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$, $\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$, and $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ consists of all matrices of the form

$$a\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + b\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} + c\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} a & c \\ c & b \end{pmatrix}.$$

This is the subspace of symmetric matrices.

Examples of subspaces of $\mathcal{M}_{2,2}(\mathbb{R})$:

- The span of $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ is the subspace of anti-symmetric matrices.

- The span of $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$, $\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$, and $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ is the subspace of upper triangular matrices.

- The span of $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$, $\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$, $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, $\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$ is the entire space $\mathcal{M}_{2,2}(\mathbb{R})$.

Spanning set

Definition. A subset S of a vector space V is called a **spanning set** for V if $\mathrm{Span}(S) = V$.

Examples.

- Vectors $\mathbf{e}_1 = (1, 0, 0)$, $\mathbf{e}_2 = (0, 1, 0)$, and $\mathbf{e}_3 = (0, 0, 1)$ form a spanning set for $\mathbb{R}^3$ as

$$(x, y, z) = x\mathbf{e}_1 + y\mathbf{e}_2 + z\mathbf{e}_3.$$

- Matrices $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$

form a spanning set for $\mathcal{M}_{2,2}(\mathbb{R})$ as

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = a\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + b\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} + c\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} + d\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}.$$

Problem Let $\mathbf{v}_1 = (1, 2, 0)$, $\mathbf{v}_2 = (3, 1, 1)$, and $\mathbf{w} = (4, -7, 3)$. Determine whether $\mathbf{w}$ belongs to $\mathrm{Span}(\mathbf{v}_1, \mathbf{v}_2)$.

We have to check if there exist $r_1, r_2 \in \mathbb{R}$ such that $\mathbf{w} = r_1\mathbf{v}_1 + r_2\mathbf{v}_2$. This vector equation is equivalent to a system of linear equations:

$$\begin{cases} 4 = r_1 + 3r_2 \\ -7 = 2r_1 + r_2 \\ 3 = 0r_1 + r_2 \end{cases} \iff \begin{cases} r_1 = -5 \\ r_2 = 3 \end{cases}$$

Thus $\mathbf{w} = -5\mathbf{v}_1 + 3\mathbf{v}_2$ is in $\mathrm{Span}(\mathbf{v}_1, \mathbf{v}_2)$.

Problem Let $\mathbf{v}_1 = (2, 5)$ and $\mathbf{v}_2 = (1, 3)$. Show that $\{\mathbf{v}_1, \mathbf{v}_2\}$ is a spanning set for $\mathbb{R}^2$.

Take any vector $\mathbf{w} = (a, b) \in \mathbb{R}^2$. We have to check that there exist $r_1, r_2 \in \mathbb{R}$ such that

$$\mathbf{w} = r_1\mathbf{v}_1 + r_2\mathbf{v}_2 \iff \begin{cases} 2r_1 + r_2 = a \\ 5r_1 + 3r_2 = b \end{cases}$$

Coefficient matrix: $C = \begin{pmatrix} 2 & 1 \\ 5 & 3 \end{pmatrix}$. $\det C = 1 \neq 0$.

Since the matrix C is invertible, the system has a unique solution for any a and b.
Thus $\mathrm{Span}(\mathbf{v}_1, \mathbf{v}_2) = \mathbb{R}^2$.

Problem Let $\mathbf{v}_1 = (2,5)$ and $\mathbf{v}_2 = (1,3)$. Show that $\{\mathbf{v}_1, \mathbf{v}_2\}$ is a spanning set for $\mathbb{R}^2$.

Alternative solution: First let us show that vectors $\mathbf{e}_1 = (1,0)$ and $\mathbf{e}_2 = (0,1)$ belong to $\mathrm{Span}(\mathbf{v}_1, \mathbf{v}_2)$.

$$\mathbf{e}_1 = r_1\mathbf{v}_1 + r_2\mathbf{v}_2 \iff \begin{cases} 2r_1 + r_2 = 1 \\ 5r_1 + 3r_2 = 0 \end{cases} \iff \begin{cases} r_1 = 3 \\ r_2 = -5 \end{cases}$$

$$\mathbf{e}_2 = r_1\mathbf{v}_1 + r_2\mathbf{v}_2 \iff \begin{cases} 2r_1 + r_2 = 0 \\ 5r_1 + 3r_2 = 1 \end{cases} \iff \begin{cases} r_1 = -1 \\ r_2 = 2 \end{cases}$$

Thus $\mathbf{e}_1 = 3\mathbf{v}_1 - 5\mathbf{v}_2$ and $\mathbf{e}_2 = -\mathbf{v}_1 + 2\mathbf{v}_2$.
Then for any vector $\mathbf{w} = (a,b) \in \mathbb{R}^2$ we have

$$\mathbf{w} = a\mathbf{e}_1 + b\mathbf{e}_2 = a(3\mathbf{v}_1 - 5\mathbf{v}_2) + b(-\mathbf{v}_1 + 2\mathbf{v}_2)$$
$$= (3a - b)\mathbf{v}_1 + (-5a + 2b)\mathbf{v}_2.$$

Problem Let $\mathbf{v}_1 = (2,5)$ and $\mathbf{v}_2 = (1,3)$. Show that $\{\mathbf{v}_1, \mathbf{v}_2\}$ is a spanning set for $\mathbb{R}^2$.

Remarks on the alternative solution:
Notice that $\mathbb{R}^2$ is spanned by vectors $\mathbf{e}_1 = (1,0)$ and $\mathbf{e}_2 = (0,1)$ since $(a,b) = a\mathbf{e}_1 + b\mathbf{e}_2$.

This is why we have checked that vectors $\mathbf{e}_1$ and $\mathbf{e}_2$ belong to $\mathrm{Span}(\mathbf{v}_1, \mathbf{v}_2)$. Then

$$\mathbf{e}_1, \mathbf{e}_2 \in \mathrm{Span}(\mathbf{v}_1, \mathbf{v}_2) \implies \mathrm{Span}(\mathbf{e}_1, \mathbf{e}_2) \subset \mathrm{Span}(\mathbf{v}_1, \mathbf{v}_2)$$
$$\implies \mathbb{R}^2 \subset \mathrm{Span}(\mathbf{v}_1, \mathbf{v}_2) \implies \mathrm{Span}(\mathbf{v}_1, \mathbf{v}_2) = \mathbb{R}^2.$$

In general, to show that $\mathrm{Span}(S_1) = \mathrm{Span}(S_2)$, it is enough to check that $S_1 \subset \mathrm{Span}(S_2)$ and $S_2 \subset \mathrm{Span}(S_1)$.

More properties of span

Let S_0 and S be subsets of a vector space V.

- $S_0 \subset S \implies \mathrm{Span}(S_0) \subset \mathrm{Span}(S)$.
- $\mathrm{Span}(S_0) = V$ and $S_0 \subset S \implies \mathrm{Span}(S) = V$.
- If $\mathbf{v}_0, \mathbf{v}_1, \dots, \mathbf{v}_k$ is a spanning set for V and $\mathbf{v}_0$ is a linear combination of vectors $\mathbf{v}_1, \dots, \mathbf{v}_k$ then $\mathbf{v}_1, \dots, \mathbf{v}_k$ is also a spanning set for V.

Indeed, if $\mathbf{v}_0 = r_1\mathbf{v}_1 + \cdots + r_k\mathbf{v}_k$, then
$t_0\mathbf{v}_0 + t_1\mathbf{v}_1 + \cdots + t_k\mathbf{v}_k = (t_0 r_1 + t_1)\mathbf{v}_1 + \cdots + (t_0 r_k + t_k)\mathbf{v}_k$.

- $\mathrm{Span}(S_0 \cup \{\mathbf{v}_0\}) = \mathrm{Span}(S_0)$ if and only if $\mathbf{v}_0 \in \mathrm{Span}(S_0)$.

If $\mathbf{v}_0 \in \mathrm{Span}(S_0)$, then $S_0 \cup \mathbf{v}_0 \subset \mathrm{Span}(S_0)$, which implies $\mathrm{Span}(S_0 \cup \{\mathbf{v}_0\}) \subset \mathrm{Span}(S_0)$. On the other hand, $\mathrm{Span}(S_0) \subset \mathrm{Span}(S_0 \cup \{\mathbf{v}_0\})$.