MATH 323

Linear Algebra

Lecture 10:
Spanning set (continued).
Linear independence.

Spanning set

Let S be a subset of a vector space V.

Definition. The **span** of the set S is the smallest subspace $W \subset V$ that contains S. If S is not empty then $W = \mathrm{Span}(S)$ consists of all linear combinations $\boxed{r_1\mathbf{v}_1 + r_2\mathbf{v}_2 + \cdots + r_k\mathbf{v}_k}$ such that $\mathbf{v}_1, \ldots, \mathbf{v}_k \in S$ and $r_1, \ldots, r_k \in \mathbb{R}$.

We say that the set S **spans** the subspace W or that S is a **spanning set** for W.

Properties of spanning sets

Let S_0 and S be subsets of a vector space V.

- $S_0 \subset S \implies \mathrm{Span}(S_0) \subset \mathrm{Span}(S)$.
- $\mathrm{Span}(S_0) = V$ and $S_0 \subset S \implies \mathrm{Span}(S) = V$.
- If $\mathbf{v}_0, \mathbf{v}_1, \dots, \mathbf{v}_k$ is a spanning set for V and $\mathbf{v}_0$ is a linear combination of vectors $\mathbf{v}_1, \dots, \mathbf{v}_k$ then $\mathbf{v}_1, \dots, \mathbf{v}_k$ is also a spanning set for V.

Indeed, if $\mathbf{v}_0 = r_1\mathbf{v}_1 + \cdots + r_k\mathbf{v}_k$, then
$t_0\mathbf{v}_0 + t_1\mathbf{v}_1 + \cdots + t_k\mathbf{v}_k = (t_0 r_1 + t_1)\mathbf{v}_1 + \cdots + (t_0 r_k + t_k)\mathbf{v}_k$.

- $\mathrm{Span}(S_0 \cup \{\mathbf{v}_0\}) = \mathrm{Span}(S_0)$ if and only if $\mathbf{v}_0 \in \mathrm{Span}(S_0)$.

"Only if" is obvious. If $\mathbf{v}_0 \in \mathrm{Span}(S_0)$, then
$S_0 \cup \{\mathbf{v}_0\} \subset \mathrm{Span}(S_0)$, hence $\mathrm{Span}(S_0 \cup \{\mathbf{v}_0\}) \subset \mathrm{Span}(S_0)$.
On the other hand, $\mathrm{Span}(S_0) \subset \mathrm{Span}(S_0 \cup \{\mathbf{v}_0\})$.

Linear independence

Definition. Let V be a vector space. Vectors $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_k \in V$ are called **linearly dependent** if they satisfy a relation

$$r_1\mathbf{v}_1 + r_2\mathbf{v}_2 + \cdots + r_k\mathbf{v}_k = \mathbf{0},$$

where the coefficients $r_1, \dots, r_k \in \mathbb{R}$ are not all equal to zero. Otherwise vectors $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_k$ are called **linearly independent**. That is, if

$$r_1\mathbf{v}_1+r_2\mathbf{v}_2+\cdots+r_k\mathbf{v}_k = \mathbf{0} \implies r_1 = \cdots = r_k = 0.$$

A set $S \subset V$ is **linearly dependent** if one can find some distinct linearly dependent vectors $\mathbf{v}_1, \dots, \mathbf{v}_k$ in S. Otherwise S is **linearly independent**.

Examples of linear independence

- Vectors $\mathbf{e}_1 = (1,0,0)$, $\mathbf{e}_2 = (0,1,0)$, and $\mathbf{e}_3 = (0,0,1)$ in $\mathbb{R}^3$.

$x\mathbf{e}_1 + y\mathbf{e}_2 + z\mathbf{e}_3 = \mathbf{0} \implies (x,y,z) = \mathbf{0}$
$\implies x = y = z = 0$

- Matrices $E_{11} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$, $E_{12} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, $E_{21} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$, and $E_{22} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$.

$aE_{11} + bE_{12} + cE_{21} + dE_{22} = O \implies \begin{pmatrix} a & b \\ c & d \end{pmatrix} = O$
$\implies a = b = c = d = 0$

Examples of linear independence

- Polynomials $1, x, x^2, \dots, x^n$.

$a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n = 0$ identically
$\implies a_i = 0$ for $0 \le i \le n$

- The infinite set $\{1, x, x^2, \dots, x^n, \dots\}$.

- Polynomials $p_1(x) = 1$, $p_2(x) = x - 1$, and $p_3(x) = (x-1)^2$.

$a_1 p_1(x) + a_2 p_2(x) + a_3 p_3(x) = a_1 + a_2(x-1) + a_3(x-1)^2 =$
$= (a_1 - a_2 + a_3) + (a_2 - 2a_3)x + a_3 x^2$.
Hence $a_1 p_1(x) + a_2 p_2(x) + a_3 p_3(x) = 0$ identically
$\implies a_1 - a_2 + a_3 = a_2 - 2a_3 = a_3 = 0$
$\implies a_1 = a_2 = a_3 = 0$

Problem Let $\mathbf{v}_1 = (1,2,0)$, $\mathbf{v}_2 = (3,1,1)$, and $\mathbf{v}_3 = (4,-7,3)$. Determine whether vectors $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ are linearly independent.

We have to check if there exist $r_1, r_2, r_3 \in \mathbb{R}$ not all zero such that $r_1\mathbf{v}_1 + r_2\mathbf{v}_2 + r_3\mathbf{v}_3 = \mathbf{0}$.

This vector equation is equivalent to a system

$$\begin{cases} r_1 + 3r_2 + 4r_3 = 0 \\ 2r_1 + r_2 - 7r_3 = 0 \\ 0r_1 + r_2 + 3r_3 = 0 \end{cases} \qquad \left(\begin{array}{ccc|c} 1 & 3 & 4 & 0 \\ 2 & 1 & -7 & 0 \\ 0 & 1 & 3 & 0 \end{array}\right)$$

The vectors $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ are linearly dependent if and only if the matrix $A = (\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3)$ is singular.
We obtain that $\det A = 0$.

Theorem The following conditions are equivalent:
(i) vectors $\mathbf{v}_1, \ldots, \mathbf{v}_k$ are linearly dependent;
(ii) one of vectors $\mathbf{v}_1, \ldots, \mathbf{v}_k$ is a linear combination of the other $k-1$ vectors.

Proof: **(i)** $\Longrightarrow$ **(ii)** Suppose that

$$r_1\mathbf{v}_1 + r_2\mathbf{v}_2 + \cdots + r_k\mathbf{v}_k = \mathbf{0},$$

where $r_i \neq 0$ for some $1 \leq i \leq k$. Then

$$\mathbf{v}_i = -\frac{r_1}{r_i}\mathbf{v}_1 - \cdots - \frac{r_{i-1}}{r_i}\mathbf{v}_{i-1} - \frac{r_{i+1}}{r_i}\mathbf{v}_{i+1} - \cdots - \frac{r_k}{r_i}\mathbf{v}_k.$$

(ii) $\Longrightarrow$ **(i)** Suppose that

$$\mathbf{v}_i = s_1\mathbf{v}_1 + \cdots + s_{i-1}\mathbf{v}_{i-1} + s_{i+1}\mathbf{v}_{i+1} + \cdots + s_k\mathbf{v}_k$$

for some scalars s_j. Then

$$s_1\mathbf{v}_1 + \cdots + s_{i-1}\mathbf{v}_{i-1} - \mathbf{v}_i + s_{i+1}\mathbf{v}_{i+1} + \cdots + s_k\mathbf{v}_k = \mathbf{0}.$$

Let A be an $n \times m$ matrix. Consider a matrix equation $A\mathbf{x} = \mathbf{0}$, where $\mathbf{0}$ is the zero column vector (of dimension n) and $\mathbf{x}$ is an unknown column vector (of dimension m).

Theorem The equation $A\mathbf{x} = \mathbf{0}$ admits a nonzero solution if and only if the columns of the matrix A are linearly dependent vectors.

Proof: Let $A = (a_{ij})$ and $\mathbf{x} = (x_1, x_2, \dots, x_m)^T$. Then

$$A\mathbf{x} = \mathbf{0} \iff \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1m} \\ a_{21} & a_{22} & \dots & a_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nm} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_m \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

$$\iff x_1 \begin{pmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{n1} \end{pmatrix} + x_2 \begin{pmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{n2} \end{pmatrix} + \dots + x_m \begin{pmatrix} a_{1m} \\ a_{2m} \\ \vdots \\ a_{nm} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}.$$

Corollary 1 Columns of a square matrix A are linearly dependent if and only if $\det A = 0$.

Idea of the proof: The equation $A\mathbf{x} = \mathbf{0}$ has a unique solution if and only if A is invertible.

Corollary 2 Rows of a square matrix A are linearly dependent if and only if $\det A = 0$.

Proof: Rows of A are columns of the transpose matrix A^T. By Corollary 1, they are linearly dependent if and only if $\det A^T = 0$. It remains to notice that $\det A = \det A^T$.

Corollary 3 Vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_m \in \mathbb{R}^n$ are linearly dependent whenever $m > n$ (i.e., the number of coordinates is less than the number of vectors).

Idea of the proof: Let A be a matrix whose columns are vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_m$. The matrix equation $A\mathbf{x} = \mathbf{0}$ is equivalent to a system of n linear homogeneous equations in m variables. Since $m > n$, there is a free variable in the system.

Example. Consider vectors $\mathbf{v}_1 = (1, -1, 1)$, $\mathbf{v}_2 = (1, 0, 0)$, $\mathbf{v}_3 = (1, 1, 1)$, and $\mathbf{v}_4 = (1, 2, 4)$ in $\mathbb{R}^3$.

Two vectors are linearly dependent if and only if they are parallel. Hence $\mathbf{v}_1$ and $\mathbf{v}_2$ are linearly independent.

Vectors $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ are linearly independent if and only if the matrix $A = (\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3)$ is invertible.

$$\det A = \begin{vmatrix} 1 & 1 & 1 \\ -1 & 0 & 1 \\ 1 & 0 & 1 \end{vmatrix} = - \begin{vmatrix} -1 & 1 \\ 1 & 1 \end{vmatrix} = 2 \neq 0.$$

Therefore $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ are linearly independent.

Four vectors in $\mathbb{R}^3$ are always linearly dependent. Thus $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4$ are linearly dependent.

Problem. Let $A = \begin{pmatrix} -1 & 1 \\ -1 & 0 \end{pmatrix}$. Determine whether matrices A, A^2, and A^3 are linearly independent.

We have $A = \begin{pmatrix} -1 & 1 \\ -1 & 0 \end{pmatrix}$, $A^2 = \begin{pmatrix} 0 & -1 \\ 1 & -1 \end{pmatrix}$, $A^3 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$.

The task is to check if there exist $r_1, r_2, r_3 \in \mathbb{R}$ not all zero such that $r_1 A + r_2 A^2 + r_3 A^3 = O$.

This matrix equation is equivalent to a system

$$\begin{cases} -r_1 + 0r_2 + r_3 = 0 \\ r_1 - r_2 + 0r_3 = 0 \\ -r_1 + r_2 + 0r_3 = 0 \\ 0r_1 - r_2 + r_3 = 0 \end{cases} \qquad \left(\begin{array}{rrr|r} -1 & 0 & 1 & 0 \\ 1 & -1 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \end{array}\right) \to \left(\begin{array}{rrr|r} 1 & -1 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

The row echelon form of the augmented matrix shows there is a free variable. Hence the system has a nonzero solution so that the matrices are linearly dependent (one relation is $A + A^2 + A^3 = O$).

More facts on linear independence

Let S_0 and S be subsets of a vector space V.

- If $S_0 \subset S$ and S is linearly independent, then so is S_0.
- If $S_0 \subset S$ and S_0 is linearly dependent, then so is S.
- If S is linearly independent in V and V is a subspace of W, then S is linearly independent in W.
- The empty set is linearly independent.
- Any set containing $\mathbf{0}$ is linearly dependent.
- Two vectors $\mathbf{v}_1$ and $\mathbf{v}_2$ are linearly dependent if and only if one of them is a scalar multiple the other.
- Two nonzero vectors $\mathbf{v}_1$ and $\mathbf{v}_2$ are linearly dependent if and only if either of them is a scalar multiple the other.
- If S_0 is linearly independent and $\mathbf{v}_0 \in V \setminus S_0$ then $S_0 \cup \{\mathbf{v}_0\}$ is linearly independent if and only if $\mathbf{v}_0 \notin \mathrm{Span}(S)$.

Problem. Show that functions e^x, e^{2x}, and e^{3x} are linearly independent in $C^\infty(\mathbb{R})$.

Suppose that $ae^x + be^{2x} + ce^{3x} = 0$ for all $x \in \mathbb{R}$, where a, b, c are constants. We have to show that $a = b = c = 0$. Differentiate this identity twice:

$$\begin{aligned} ae^x + be^{2x} + ce^{3x} &= 0, \\ ae^x + 2be^{2x} + 3ce^{3x} &= 0, \\ ae^x + 4be^{2x} + 9ce^{3x} &= 0. \end{aligned}$$

It follows that $A(x)\mathbf{v} = \mathbf{0}$, where

$$A(x) = \begin{pmatrix} e^x & e^{2x} & e^{3x} \\ e^x & 2e^{2x} & 3e^{3x} \\ e^x & 4e^{2x} & 9e^{3x} \end{pmatrix}, \quad \mathbf{v} = \begin{pmatrix} a \\ b \\ c \end{pmatrix}.$$

$$A(x) = \begin{pmatrix} e^x & e^{2x} & e^{3x} \\ e^x & 2e^{2x} & 3e^{3x} \\ e^x & 4e^{2x} & 9e^{3x} \end{pmatrix}, \quad \mathbf{v} = \begin{pmatrix} a \\ b \\ c \end{pmatrix}.$$

$$\det A(x) = e^x \begin{vmatrix} 1 & e^{2x} & e^{3x} \\ 1 & 2e^{2x} & 3e^{3x} \\ 1 & 4e^{2x} & 9e^{3x} \end{vmatrix} = e^x e^{2x} \begin{vmatrix} 1 & 1 & e^{3x} \\ 1 & 2 & 3e^{3x} \\ 1 & 4 & 9e^{3x} \end{vmatrix}$$

$$= e^x e^{2x} e^{3x} \begin{vmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 4 & 9 \end{vmatrix} = e^{6x} \begin{vmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 4 & 9 \end{vmatrix} = e^{6x} \begin{vmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 1 & 4 & 9 \end{vmatrix}$$

$$= e^{6x} \begin{vmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 3 & 8 \end{vmatrix} = e^{6x} \begin{vmatrix} 1 & 2 \\ 3 & 8 \end{vmatrix} = 2e^{6x} \neq 0.$$

Since the matrix $A(x)$ is invertible, we obtain

$A(x)\mathbf{v} = \mathbf{0} \implies \mathbf{v} = \mathbf{0} \implies a = b = c = 0$

Wronskian

Let $f_1, f_2, \ldots, f_n$ be smooth functions on an interval $[a, b]$. The **Wronskian** $W[f_1, f_2, \ldots, f_n]$ is a function on $[a, b]$ defined by

$$W[f_1, f_2, \ldots, f_n](x) = \begin{vmatrix} f_1(x) & f_2(x) & \cdots & f_n(x) \\ f_1'(x) & f_2'(x) & \cdots & f_n'(x) \\ \vdots & \vdots & \ddots & \vdots \\ f_1^{(n-1)}(x) & f_2^{(n-1)}(x) & \cdots & f_n^{(n-1)}(x) \end{vmatrix}.$$

Theorem If $W[f_1, f_2, \ldots, f_n](x_0) \neq 0$ for some $x_0 \in [a, b]$ then the functions $f_1, f_2, \ldots, f_n$ are linearly independent in $C[a, b]$.

Theorem Let $\lambda_1, \lambda_2, \ldots, \lambda_k$ be distinct real numbers. Then the functions $e^{\lambda_1 x}, e^{\lambda_2 x}, \ldots, e^{\lambda_k x}$ are linearly independent.

$$W[e^{\lambda_1 x}, e^{\lambda_2 x}, \ldots, e^{\lambda_k x}](x) = \begin{vmatrix} e^{\lambda_1 x} & e^{\lambda_2 x} & \cdots & e^{\lambda_k x} \\ \lambda_1 e^{\lambda_1 x} & \lambda_2 e^{\lambda_2 x} & \cdots & \lambda_k e^{\lambda_k x} \\ \vdots & \vdots & \ddots & \vdots \\ \lambda_1^{k-1} e^{\lambda_1 x} & \lambda_2^{k-1} e^{\lambda_2 x} & \cdots & \lambda_k^{k-1} e^{\lambda_k x} \end{vmatrix}$$

$$= e^{(\lambda_1 + \lambda_2 + \cdots + \lambda_k)x} \begin{vmatrix} 1 & 1 & \cdots & 1 \\ \lambda_1 & \lambda_2 & \cdots & \lambda_k \\ \vdots & \vdots & \ddots & \vdots \\ \lambda_1^{k-1} & \lambda_2^{k-1} & \cdots & \lambda_k^{k-1} \end{vmatrix} \neq 0$$

since the latter determinant is the transpose of the Vandermonde determinant.