MATH 323

Linear Algebra

Lecture 11:

Wronskian.

Basis of a vector space.

Dimension.

Linear independence

Definition. Let V be a vector space. Vectors $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_k \in V$ are called **linearly dependent** if they satisfy a relation

$$r_1\mathbf{v}_1 + r_2\mathbf{v}_2 + \cdots + r_k\mathbf{v}_k = \mathbf{0},$$

where the coefficients $r_1, \dots, r_k \in \mathbb{R}$ are not all equal to zero. Otherwise vectors $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_k$ are called **linearly independent**. That is, if

$$r_1\mathbf{v}_1 + r_2\mathbf{v}_2 + \cdots + r_k\mathbf{v}_k = \mathbf{0} \implies r_1 = \cdots = r_k = 0.$$

A set $S \subset V$ is **linearly dependent** if one can find some distinct linearly dependent vectors $\mathbf{v}_1, \dots, \mathbf{v}_k$ in S. Otherwise S is **linearly independent**.

Theorem Vectors $\mathbf{v}_1, \ldots, \mathbf{v}_k \in V$ are linearly dependent if and only if one of them is a linear combination of the other $k-1$ vectors.

Examples of linear independence.

- Vectors $\mathbf{e}_1 = (1,0,0)$, $\mathbf{e}_2 = (0,1,0)$, and $\mathbf{e}_3 = (0,0,1)$ in $\mathbb{R}^3$.
- Matrices $E_{11} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$, $E_{12} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, $E_{21} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$, and $E_{22} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$.
- Polynomials $1, x, x^2, \ldots, x^n, \ldots$

Wronskian

Let $f_1, f_2, \dots, f_n$ be smooth functions on an interval $[a, b]$. The **Wronskian** $W[f_1, f_2, \dots, f_n]$ is a function on $[a, b]$ defined by

$$W[f_1, f_2, \dots, f_n](x) = \begin{vmatrix} f_1(x) & f_2(x) & \cdots & f_n(x) \\ f_1'(x) & f_2'(x) & \cdots & f_n'(x) \\ \vdots & \vdots & \ddots & \vdots \\ f_1^{(n-1)}(x) & f_2^{(n-1)}(x) & \cdots & f_n^{(n-1)}(x) \end{vmatrix}.$$

Theorem If $W[f_1, f_2, \dots, f_n](x_0) \neq 0$ for some $x_0 \in [a, b]$ then the functions $f_1, f_2, \dots, f_n$ are linearly independent in $C[a, b]$.

Proof: Assume that $r_1 f_1(x) + r_2 f_2(x) + \cdots + r_n f_n(x) = 0$ for all $x \in [a, b]$, where $r_1, r_2, \ldots, r_n$ are constants. We have to show that $r_i = 0$, $1 \le i \le n$.

Differentiating this identity $n-1$ times, we get $n-1$ more identities

$$r_1 f_1'(x) + r_2 f_2'(x) + \cdots + r_n f_n'(x) = 0,$$
$$\cdots\cdots\cdots\cdots$$
$$r_1 f_1^{(n-1)}(x) + r_2 f_2^{(n-1)}(x) + \cdots + r_n f_n^{(n-1)}(x) = 0.$$

We can consider these n identities as a system of linear homogeneous equations (depending on the parameter x) in variables $r_1, r_2, \ldots, r_n$. Note that we need a solution common for all values of the parameter.

By construction, $W[f_1, f_2, \ldots, f_n](x) = \det A(x)$, where $A(x)$ is the coefficient matrix of the system. Since $\det A(x_0) \neq 0$, the matrix $A(x_0)$ is invertible. Hence $r_1 = r_2 = \cdots = r_n = 0$ is the only solution of the system for $x = x_0$ (let alone all $x \in [a, b]$).

Theorem Let $\lambda_1, \lambda_2, \ldots, \lambda_k$ be distinct real numbers. Then the functions $e^{\lambda_1 x}, e^{\lambda_2 x}, \ldots, e^{\lambda_k x}$ are linearly independent.

$$W[e^{\lambda_1 x}, e^{\lambda_2 x}, \ldots, e^{\lambda_k x}](x) = \begin{vmatrix} e^{\lambda_1 x} & e^{\lambda_2 x} & \cdots & e^{\lambda_k x} \\ \lambda_1 e^{\lambda_1 x} & \lambda_2 e^{\lambda_2 x} & \cdots & \lambda_k e^{\lambda_k x} \\ \vdots & \vdots & \ddots & \vdots \\ \lambda_1^{k-1} e^{\lambda_1 x} & \lambda_2^{k-1} e^{\lambda_2 x} & \cdots & \lambda_k^{k-1} e^{\lambda_k x} \end{vmatrix}$$

$$= e^{(\lambda_1 + \lambda_2 + \cdots + \lambda_k)x} \begin{vmatrix} 1 & 1 & \cdots & 1 \\ \lambda_1 & \lambda_2 & \cdots & \lambda_k \\ \vdots & \vdots & \ddots & \vdots \\ \lambda_1^{k-1} & \lambda_2^{k-1} & \cdots & \lambda_k^{k-1} \end{vmatrix} \neq 0$$

since the latter determinant is the transpose of the Vandermonde determinant.

Basis

Definition. Let V be a vector space. Any linearly independent spanning set for V is called a **basis**.

Suppose that a nonempty set $S \subset V$ is a basis for V. "Spanning set" means that any vector $\mathbf{v} \in V$ can be represented as a linear combination

$$\mathbf{v} = r_1\mathbf{v}_1 + r_2\mathbf{v}_2 + \cdots + r_k\mathbf{v}_k,$$

where $\mathbf{v}_1, \dots, \mathbf{v}_k$ are distinct vectors from S and $r_1, \dots, r_k \in \mathbb{R}$. "Linearly independent" implies that the above representation is unique:

$$\mathbf{v} = r_1\mathbf{v}_1 + r_2\mathbf{v}_2 + \cdots + r_k\mathbf{v}_k = r_1'\mathbf{v}_1 + r_2'\mathbf{v}_2 + \cdots + r_k'\mathbf{v}_k$$
$$\Longrightarrow \quad (r_1 - r_1')\mathbf{v}_1 + (r_2 - r_2')\mathbf{v}_2 + \cdots + (r_k - r_k')\mathbf{v}_k = \mathbf{0}$$
$$\Longrightarrow \quad r_1 - r_1' = r_2 - r_2' = \ldots = r_k - r_k' = 0$$

Examples. • Standard basis for $\mathbb{R}^n$:
$\mathbf{e}_1 = (1, 0, 0, \ldots, 0, 0)$, $\mathbf{e}_2 = (0, 1, 0, \ldots, 0, 0)$, $\ldots$,
$\mathbf{e}_n = (0, 0, 0, \ldots, 0, 1)$.
Indeed, $(x_1, x_2, \ldots, x_n) = x_1\mathbf{e}_1 + x_2\mathbf{e}_2 + \cdots + x_n\mathbf{e}_n$.

• Matrices $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$, $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, $\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$, $\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$
form a basis for $\mathcal{M}_{2,2}(\mathbb{R})$.

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = a\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + b\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} + c\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} + d\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}.$$

• Polynomials $1, x, x^2, \ldots, x^{n-1}$ form a basis for
$\mathcal{P}_n = \{a_0 + a_1x + \cdots + a_{n-1}x^{n-1} : a_i \in \mathbb{R}\}$.

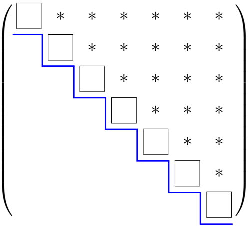
• The infinite set $\{1, x, x^2, \ldots, x^n, \ldots\}$ is a basis for $\mathcal{P}$, the space of all polynomials.

Let $\mathbf{v}, \mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_k \in \mathbb{R}^n$ and $r_1, r_2, \dots, r_k \in \mathbb{R}$. The vector equation $r_1\mathbf{v}_1 + r_2\mathbf{v}_2 + \cdots + r_k\mathbf{v}_k = \mathbf{v}$ is equivalent to the matrix equation $A\mathbf{x} = \mathbf{v}$, where

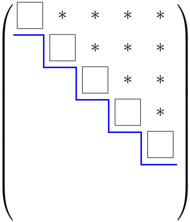
$$A = (\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_k), \qquad \mathbf{x} = \begin{pmatrix} r_1 \\ \vdots \\ r_k \end{pmatrix}.$$

That is, A is the $n \times k$ matrix such that vectors $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_k$ are consecutive columns of A.

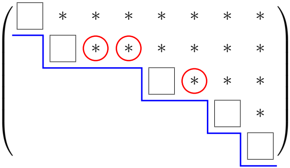
- *Vectors $\mathbf{v}_1, \dots, \mathbf{v}_k$ span $\mathbb{R}^n$ if the row echelon form of A has no zero rows (consistency).*
- *Vectors $\mathbf{v}_1, \dots, \mathbf{v}_k$ are linearly independent if the row echelon form of A has a leading entry in each column (no free variables).*



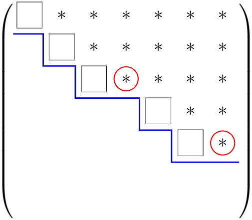
spanning
linear independence



no spanning
linear independence



spanning
no linear independence



no spanning
no linear independence

Bases for $\mathbb{R}^n$

Let $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_k$ be vectors in $\mathbb{R}^n$.

Theorem 1 If $k < n$ then the vectors $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_k$ do not span $\mathbb{R}^n$.

Theorem 2 If $k > n$ then the vectors $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_k$ are linearly dependent.

Theorem 3 If $k = n$ then the following conditions are equivalent:
(i) $\{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n\}$ is a basis for $\mathbb{R}^n$;
(ii) $\{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n\}$ is a spanning set for $\mathbb{R}^n$;
(iii) $\{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n\}$ is a linearly independent set.

Example. Consider vectors $\mathbf{v}_1 = (1, -1, 1)$, $\mathbf{v}_2 = (1, 0, 0)$, $\mathbf{v}_3 = (1, 1, 1)$, and $\mathbf{v}_4 = (1, 2, 4)$ in $\mathbb{R}^3$.

Vectors $\mathbf{v}_1$ and $\mathbf{v}_2$ are linearly independent (as they are not parallel), but they do not span $\mathbb{R}^3$.

Vectors $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ are linearly independent since

$$\begin{vmatrix} 1 & 1 & 1 \\ -1 & 0 & 1 \\ 1 & 0 & 1 \end{vmatrix} = -\begin{vmatrix} -1 & 1 \\ 1 & 1 \end{vmatrix} = -(-2) = 2 \neq 0.$$

Therefore $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is a basis for $\mathbb{R}^3$.

Vectors $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4$ span $\mathbb{R}^3$ (because $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ already span $\mathbb{R}^3$), but they are linearly dependent.

Dimension

Theorem 1 Any vector space has a basis.

Theorem 2 If a vector space V has a finite basis, then all bases for V are finite and have the same number of elements.

Definition. The **dimension** of a vector space V, denoted $\dim V$, is the number of elements in any of its bases.

Examples. • $\dim \mathbb{R}^n = n$

• $\mathcal{M}_{2,2}(\mathbb{R})$: the space of 2×2 matrices
$\dim \mathcal{M}_{2,2}(\mathbb{R}) = 4$

• $\mathcal{M}_{m,n}(\mathbb{R})$: the space of $m\times n$ matrices
$\dim \mathcal{M}_{m,n}(\mathbb{R}) = mn$

• $\mathcal{P}_n$: polynomials of degree less than n
$\dim \mathcal{P}_n = n$

• $\mathcal{P}$: the space of all polynomials
$\dim \mathcal{P} = \infty$

• $\{\mathbf{0}\}$: the trivial vector space
$\dim \{\mathbf{0}\} = 0$

Problem. Find the dimension of the plane $x + 2z = 0$ in $\mathbb{R}^3$.

The general solution of the equation $x + 2z = 0$ is

$$\begin{cases} x = -2s \\ y = t \\ z = s \end{cases} \qquad (t, s \in \mathbb{R})$$

That is, $(x, y, z) = (-2s, t, s) = t(0, 1, 0) + s(-2, 0, 1)$.

Hence the plane is the span of vectors $\mathbf{v}_1 = (0, 1, 0)$ and $\mathbf{v}_2 = (-2, 0, 1)$. These vectors are linearly independent as they are not parallel.

Thus $\{\mathbf{v}_1, \mathbf{v}_2\}$ is a basis so that the dimension of the plane is 2.

How to find a basis?

Theorem Let S be a subset of a vector space V. Then the following conditions are equivalent:

(i) S is a linearly independent spanning set for V, i.e., a basis;

(ii) S is a minimal spanning set for V;

(iii) S is a maximal linearly independent subset of V.

"Minimal spanning set" means "remove any element from this set, and it is no longer a spanning set".

"Maximal linearly independent subset" means "add any element of V to this set, and it will become linearly dependent".

Theorem Let V be a vector space. Then

(i) any spanning set for V can be reduced to a minimal spanning set;

(ii) any linearly independent subset of V can be extended to a maximal linearly independent set.

Corollary 1 Any spanning set contains a basis while any linearly independent set is contained in a basis.

Corollary 2 A vector space is finite-dimensional if and only if it is spanned by a finite set.

How to find a basis?

Approach 1. Get a spanning set for the vector space, then reduce this set to a basis dropping one vector at a time.

Proposition Let $\mathbf{v}_0, \mathbf{v}_1, \dots, \mathbf{v}_k$ be a spanning set for a vector space V. If $\mathbf{v}_0$ is a linear combination of vectors $\mathbf{v}_1, \dots, \mathbf{v}_k$ then $\mathbf{v}_1, \dots, \mathbf{v}_k$ is also a spanning set for V.

Indeed, if $\mathbf{v}_0 = r_1\mathbf{v}_1 + \cdots + r_k\mathbf{v}_k$, then

$$t_0\mathbf{v}_0 + t_1\mathbf{v}_1 + \cdots + t_k\mathbf{v}_k =$$
$$= (t_0 r_1 + t_1)\mathbf{v}_1 + \cdots + (t_0 r_k + t_k)\mathbf{v}_k.$$

How to find a basis?

Approach 2. Build a maximal linearly independent set adding one vector at a time.

If the vector space V is trivial, it has the empty basis. If $V \neq \{\mathbf{0}\}$, pick any vector $\mathbf{v}_1 \neq \mathbf{0}$. If $\mathbf{v}_1$ spans V, it is a basis. Otherwise pick any vector $\mathbf{v}_2 \in V$ that is not in the span of $\mathbf{v}_1$. If $\mathbf{v}_1$ and $\mathbf{v}_2$ span V, they constitute a basis. Otherwise pick any vector $\mathbf{v}_3 \in V$ that is not in the span of $\mathbf{v}_1$ and $\mathbf{v}_2$. And so on...

Modifications. Instead of the empty set, we can start with any linearly independent set (if we are given one). If we are given a spanning set S, it is enough to pick new vectors only in S.

Remark. This inductive procedure works for finite-dimensional vector spaces. There is an analogous procedure for infinite-dimensional spaces (*transfinite induction*).

Vectors $\mathbf{v}_1 = (0, 1, 0)$ and $\mathbf{v}_2 = (-2, 0, 1)$ are linearly independent.

Problem. Extend the set $\{\mathbf{v}_1, \mathbf{v}_2\}$ to a basis for $\mathbb{R}^3$.

Our task is to find a vector $\mathbf{v}_3$ that is not a linear combination of $\mathbf{v}_1$ and $\mathbf{v}_2$.
Then $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ will be a basis for $\mathbb{R}^3$.

Hint 1. $\mathbf{v}_1$ and $\mathbf{v}_2$ span the plane $x + 2z = 0$.

The vector $\mathbf{v}_3 = (1, 1, 1)$ does not lie in the plane $x + 2z = 0$, hence it is not a linear combination of $\mathbf{v}_1$ and $\mathbf{v}_2$. Thus $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is a basis for $\mathbb{R}^3$.

Vectors $\mathbf{v}_1 = (0, 1, 0)$ and $\mathbf{v}_2 = (-2, 0, 1)$ are linearly independent.

Problem. Extend the set $\{\mathbf{v}_1, \mathbf{v}_2\}$ to a basis for $\mathbb{R}^3$.

Our task is to find a vector $\mathbf{v}_3$ that is not a linear combination of $\mathbf{v}_1$ and $\mathbf{v}_2$. Then $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ will be a basis for $\mathbb{R}^3$.

Hint 2. Since vectors $\mathbf{e}_1 = (1, 0, 0)$, $\mathbf{e}_2 = (0, 1, 0)$, and $\mathbf{e}_3 = (0, 0, 1)$ form a spanning set for $\mathbb{R}^3$, at least one of them can be chosen as $\mathbf{v}_3$.

Let us check that $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{e}_1\}$ and $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{e}_3\}$ are two bases for $\mathbb{R}^3$:

$$\begin{vmatrix} 0 & -2 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{vmatrix} = 1 \neq 0, \qquad \begin{vmatrix} 0 & -2 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 1 \end{vmatrix} = 2 \neq 0.$$

Problem. Find a basis for the vector space V spanned by vectors $\mathbf{w}_1 = (1,1,0)$, $\mathbf{w}_2 = (0,1,1)$, $\mathbf{w}_3 = (2,3,1)$, and $\mathbf{w}_4 = (1,1,1)$.

To pare this spanning set, we need to find a relation of the form $r_1\mathbf{w}_1 + r_2\mathbf{w}_2 + r_3\mathbf{w}_3 + r_4\mathbf{w}_4 = \mathbf{0}$, where $r_i \in \mathbb{R}$ are not all equal to zero. Equivalently,

$$\begin{pmatrix} 1 & 0 & 2 & 1 \\ 1 & 1 & 3 & 1 \\ 0 & 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} r_1 \\ r_2 \\ r_3 \\ r_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

To solve this system of linear equations for r_1, r_2, r_3, r_4, we apply row reduction.

$$
\begin{pmatrix} 1 & 0 & 2 & 1 \\ 1 & 1 & 3 & 1 \\ 0 & 1 & 1 & 1 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & 2 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 \end{pmatrix} \to \begin{pmatrix} \color{red}{1} & 0 & 2 & 1 \\ 0 & \color{red}{1} & 1 & 0 \\ 0 & 0 & 0 & \color{red}{1} \end{pmatrix}
$$

$$
\to \begin{pmatrix} \color{red}{1} & 0 & 2 & 0 \\ 0 & \color{red}{1} & 1 & 0 \\ 0 & 0 & 0 & \color{red}{1} \end{pmatrix} \qquad \text{(reduced row echelon form)}
$$

$$
\begin{cases} r_1 + 2r_3 = 0 \\ r_2 + r_3 = 0 \\ r_4 = 0 \end{cases} \iff \begin{cases} r_1 = -2r_3 \\ r_2 = -r_3 \\ r_4 = 0 \end{cases}
$$

General solution: $(r_1, r_2, r_3, r_4) = (-2t, -t, t, 0)$, $t \in \mathbb{R}$.

Particular solution: $(r_1, r_2, r_3, r_4) = (2, 1, -1, 0)$.

Problem. Find a basis for the vector space V spanned by vectors $\mathbf{w}_1 = (1,1,0)$, $\mathbf{w}_2 = (0,1,1)$, $\mathbf{w}_3 = (2,3,1)$, and $\mathbf{w}_4 = (1,1,1)$.

We have obtained that $2\mathbf{w}_1 + \mathbf{w}_2 - \mathbf{w}_3 = \mathbf{0}$. Hence any of vectors $\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3$ can be dropped. For instance, $V = \mathrm{Span}(\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_4)$.

Let us check whether vectors $\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_4$ are linearly independent:

$$\begin{vmatrix} 1 & 0 & 1 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{vmatrix} = \begin{vmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 0 \end{vmatrix} = \begin{vmatrix} 1 & 1 \\ 0 & 1 \end{vmatrix} = 1 \neq 0.$$

They are!!! Thus $\{\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_4\}$ is a basis for V. Moreover, it follows that $V = \mathbb{R}^3$.