MATH 323

Linear Algebra

Lecture 13:
Review for Test 1.

Topics for Test 1

Part I: Elementary linear algebra (Leon 1.1–1.5, 2.1–2.2)

- Systems of linear equations: elementary operations, Gaussian elimination, back substitution.

- Matrix of coefficients and augmented matrix. Elementary row operations, row echelon form and reduced row echelon form.

- Matrix algebra. Inverse matrix.

- Determinants: explicit formulas for 2×2 and 3×3 matrices, row and column expansions, elementary row and column operations.

Topics for Test 1

Part II: Abstract linear algebra (Leon 3.1–3.4, 3.6)

- Vector spaces (vectors, matrices, polynomials, functional spaces).
- Subspaces. Nullspace, column space, and row space of a matrix.
- Span, spanning set.
- Linear independence.
- Basis and dimension.
- Rank and nullity of a matrix.

Proofs to know

Theorem 1 If two $n \times n$ matrices A and B are invertible, then the product AB is also invertible and $(AB)^{-1} = B^{-1}A^{-1}$.

Theorem 2 If an $n \times n$ matrix A is invertible, then for any n-dimensional column vector $\mathbf{b}$ the matrix equation $A\mathbf{x} = \mathbf{b}$ has a unique solution, which is $\mathbf{x} = A^{-1}\mathbf{b}$.

Proofs to know

Theorem 3 In any vector space, the zero vector is unique and the negative vector is unique.

Theorem 4 For any vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k$ in a vector space V, the set of all linear combinations $r_1\mathbf{v}_1 + r_2\mathbf{v}_2 + \cdots + r_k\mathbf{v}_k$, $r_i \in \mathbb{R}$ is a subspace of V.

Theorem 5 Vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k$ $(k \geq 2)$ are linearly dependent if and only if one of them is a linear combination of the other $k - 1$ vectors.

Theorem 6 Functions $f_1, f_2, \ldots, f_n \in C[a, b]$ are linearly independent whenever their Wronskian $W[f_1, f_2, \ldots, f_n]$ is well defined and not identically zero on $[a, b]$.

Sample problems for Test 1

Problem 1 (15 pts.) Find a quadratic polynomial $p(x)$ such that $p(1) = 1$, $p(2) = 3$, and $p(3) = 7$.

Problem 2 (25 pts.) Let $A = \begin{pmatrix} 1 & -2 & 4 & 1 \\ 2 & 3 & 2 & 0 \\ 2 & 0 & -1 & 1 \\ 2 & 0 & 0 & 1 \end{pmatrix}$.

(i) Evaluate the determinant of the matrix A.
(ii) Find the inverse matrix A^{-1}.

Problem 3 (20 pts.) Determine which of the following subsets of $\mathbb{R}^3$ are subspaces. Briefly explain.

(i) The set S_1 of vectors $(x, y, z) \in \mathbb{R}^3$ such that $xyz = 0$.
(ii) The set S_2 of vectors $(x, y, z) \in \mathbb{R}^3$ such that $x + y + z = 0$.
(iii) The set S_3 of vectors $(x, y, z) \in \mathbb{R}^3$ such that $y^2 + z^2 = 0$.
(iv) The set S_4 of vectors $(x, y, z) \in \mathbb{R}^3$ such that $y^2 - z^2 = 0$.

Problem 4 (30 pts.) Let $B = \begin{pmatrix} 0 & -1 & 4 & 1 \\ 1 & 1 & 2 & -1 \\ -3 & 0 & -1 & 0 \\ 2 & -1 & 0 & 1 \end{pmatrix}$.

(i) Find the rank and the nullity of the matrix B.
(ii) Find a basis for the row space of B, then extend this basis to a basis for $\mathbb{R}^4$.
(iii) Find a basis for the nullspace of B.

Bonus Problem 5 (15 pts.) Show that the functions $f_1(x) = x$, $f_2(x) = xe^x$, and $f_3(x) = e^{-x}$ are linearly independent in the vector space $C^\infty(\mathbb{R})$.

Bonus Problem 6 (15 pts.) Let V be a finite-dimensional vector space and V_0 be a proper subspace of V (where proper means that $V_0 \neq V$). Prove that $\dim V_0 < \dim V$.

Problem 1. Find a quadratic polynomial $p(x)$ such that $p(1) = 1$, $p(2) = 3$, and $p(3) = 7$.

Let $p(x) = a + bx + cx^2$. Then $p(1) = a + b + c$, $p(2) = a + 2b + 4c$, and $p(3) = a + 3b + 9c$.
The coefficients a, b, and c have to be chosen so that

$$\begin{cases} a + b + c = 1, \\ a + 2b + 4c = 3, \\ a + 3b + 9c = 7. \end{cases}$$

We solve this system of linear equations using elementary operations:

$$\begin{cases} a + b + c = 1 \\ a + 2b + 4c = 3 \\ a + 3b + 9c = 7 \end{cases} \iff \begin{cases} a + b + c = 1 \\ b + 3c = 2 \\ a + 3b + 9c = 7 \end{cases}$$

$$\iff \begin{cases} a+b+c=1 \\ b+3c=2 \\ a+3b+9c=7 \end{cases} \iff \begin{cases} a+b+c=1 \\ b+3c=2 \\ 2b+8c=6 \end{cases}$$

$$\iff \begin{cases} a+b+c=1 \\ b+3c=2 \\ b+4c=3 \end{cases} \iff \begin{cases} a+b+c=1 \\ b+3c=2 \\ c=1 \end{cases}$$

$$\iff \begin{cases} a+b+c=1 \\ b=-1 \\ c=1 \end{cases} \iff \begin{cases} a=1 \\ b=-1 \\ c=1 \end{cases}$$

Thus the desired polynomial is $p(x) = x^2 - x + 1$.

Problem 2. Let $A = \begin{pmatrix} 1 & -2 & 4 & 1 \\ 2 & 3 & 2 & 0 \\ 2 & 0 & -1 & 1 \\ 2 & 0 & 0 & 1 \end{pmatrix}$.

(i) Evaluate the determinant of the matrix A.

Subtract the 4th row of A from the 3rd row:

$$\begin{vmatrix} 1 & -2 & 4 & 1 \\ 2 & 3 & 2 & 0 \\ 2 & 0 & -1 & 1 \\ 2 & 0 & 0 & 1 \end{vmatrix} = \begin{vmatrix} 1 & -2 & 4 & 1 \\ 2 & 3 & 2 & 0 \\ 0 & 0 & -1 & 0 \\ 2 & 0 & 0 & 1 \end{vmatrix}.$$

Expand the determinant by the 3rd row:

$$\begin{vmatrix} 1 & -2 & 4 & 1 \\ 2 & 3 & 2 & 0 \\ 0 & 0 & -1 & 0 \\ 2 & 0 & 0 & 1 \end{vmatrix} = (-1) \begin{vmatrix} 1 & -2 & 1 \\ 2 & 3 & 0 \\ 2 & 0 & 1 \end{vmatrix}.$$

Expand the determinant by the 3rd column:

$$(-1) \begin{vmatrix} 1 & -2 & 1 \\ 2 & 3 & 0 \\ 2 & 0 & 1 \end{vmatrix} = (-1) \left(\begin{vmatrix} 2 & 3 \\ 2 & 0 \end{vmatrix} + \begin{vmatrix} 1 & -2 \\ 2 & 3 \end{vmatrix} \right) = -1.$$

Problem 2. Let $A = \begin{pmatrix} 1 & -2 & 4 & 1 \\ 2 & 3 & 2 & 0 \\ 2 & 0 & -1 & 1 \\ 2 & 0 & 0 & 1 \end{pmatrix}$.

(ii) Find the inverse matrix A^{-1}.

First we merge the matrix A with the identity matrix into one 4×8 matrix

$$(A \,|\, I) = \left(\begin{array}{cccc|cccc} 1 & -2 & 4 & 1 & 1 & 0 & 0 & 0 \\ 2 & 3 & 2 & 0 & 0 & 1 & 0 & 0 \\ 2 & 0 & -1 & 1 & 0 & 0 & 1 & 0 \\ 2 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \end{array}\right).$$

Then we apply elementary row operations to this matrix until the left part becomes the identity matrix.

Subtract 2 times the 1st row from the 2nd row:

$$\left(\begin{array}{cccc|cccc} 1 & -2 & 4 & 1 & 1 & 0 & 0 & 0 \\ 0 & 7 & -6 & -2 & -2 & 1 & 0 & 0 \\ 2 & 0 & -1 & 1 & 0 & 0 & 1 & 0 \\ 2 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \end{array}\right)$$

Subtract 2 times the 1st row from the 3rd row:

$$\left(\begin{array}{cccc|cccc} 1 & -2 & 4 & 1 & 1 & 0 & 0 & 0 \\ 0 & 7 & -6 & -2 & -2 & 1 & 0 & 0 \\ 0 & 4 & -9 & -1 & -2 & 0 & 1 & 0 \\ 2 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \end{array}\right)$$

Subtract 2 times the 1st row from the 4th row:

$$\left(\begin{array}{cccc|cccc} 1 & -2 & 4 & 1 & 1 & 0 & 0 & 0 \\ 0 & 7 & -6 & -2 & -2 & 1 & 0 & 0 \\ 0 & 4 & -9 & -1 & -2 & 0 & 1 & 0 \\ 0 & 4 & -8 & -1 & -2 & 0 & 0 & 1 \end{array}\right)$$

Subtract 2 times the 4th row from the 2nd row:

$$\left(\begin{array}{cccc|cccc} 1 & -2 & 4 & 1 & 1 & 0 & 0 & 0 \\ 0 & -1 & 10 & 0 & 2 & 1 & 0 & -2 \\ 0 & 4 & -9 & -1 & -2 & 0 & 1 & 0 \\ 0 & 4 & -8 & -1 & -2 & 0 & 0 & 1 \end{array}\right)$$

Subtract the 4th row from the 3rd row:

$$\left(\begin{array}{cccc|cccc} 1 & -2 & 4 & 1 & 1 & 0 & 0 & 0 \\ 0 & -1 & 10 & 0 & 2 & 1 & 0 & -2 \\ 0 & 0 & -1 & 0 & 0 & 0 & 1 & -1 \\ 0 & 4 & -8 & -1 & -2 & 0 & 0 & 1 \end{array}\right)$$

Add 4 times the 2nd row to the 4th row:

$$\left(\begin{array}{cccc|cccc} 1 & -2 & 4 & 1 & 1 & 0 & 0 & 0 \\ 0 & -1 & 10 & 0 & 2 & 1 & 0 & -2 \\ 0 & 0 & -1 & 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 32 & -1 & 6 & 4 & 0 & -7 \end{array}\right)$$

Add 32 times the 3rd row to the 4th row:

$$\left(\begin{array}{rrrr|rrrr} 1 & -2 & 4 & 1 & 1 & 0 & 0 & 0 \\ 0 & -1 & 10 & 0 & 2 & 1 & 0 & -2 \\ 0 & 0 & -1 & 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & -1 & 6 & 4 & 32 & -39 \end{array}\right)$$

Add 10 times the 3rd row to the 2nd row:

$$\left(\begin{array}{rrrr|rrrr} 1 & -2 & 4 & 1 & 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 2 & 1 & 10 & -12 \\ 0 & 0 & -1 & 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & -1 & 6 & 4 & 32 & -39 \end{array}\right)$$

Add the 4th row to the 1st row:

$$\left(\begin{array}{rrrr|rrrr} 1 & -2 & 4 & 0 & 7 & 4 & 32 & -39 \\ 0 & -1 & 0 & 0 & 2 & 1 & 10 & -12 \\ 0 & 0 & -1 & 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & -1 & 6 & 4 & 32 & -39 \end{array}\right)$$

Add 4 times the 3rd row to the 1st row:

$$\left(\begin{array}{cccc|cccc} 1 & -2 & 0 & 0 & 7 & 4 & 36 & -43 \\ 0 & -1 & 0 & 0 & 2 & 1 & 10 & -12 \\ 0 & 0 & -1 & 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & -1 & 6 & 4 & 32 & -39 \end{array}\right)$$

Subtract 2 times the 2nd row from the 1st row:

$$\left(\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 3 & 2 & 16 & -19 \\ 0 & -1 & 0 & 0 & 2 & 1 & 10 & -12 \\ 0 & 0 & -1 & 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & -1 & 6 & 4 & 32 & -39 \end{array}\right)$$

Multiply the 2nd, the 3rd, and the 4th rows by -1:

$$\left(\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 3 & 2 & 16 & -19 \\ 0 & 1 & 0 & 0 & -2 & -1 & -10 & 12 \\ 0 & 0 & 1 & 0 & 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & 1 & -6 & -4 & -32 & 39 \end{array}\right)$$

$$\left(\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 3 & 2 & 16 & -19 \\ 0 & 1 & 0 & 0 & -2 & -1 & -10 & 12 \\ 0 & 0 & 1 & 0 & 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & 1 & -6 & -4 & -32 & 39 \end{array}\right) = (I \mid A^{-1})$$

Finally the left part of our 4×8 matrix is transformed into the identity matrix. Therefore the current right part is the inverse matrix of A. Thus

$$A^{-1} = \begin{pmatrix} 1 & -2 & 4 & 1 \\ 2 & 3 & 2 & 0 \\ 2 & 0 & -1 & 1 \\ 2 & 0 & 0 & 1 \end{pmatrix}^{-1} = \begin{pmatrix} 3 & 2 & 16 & -19 \\ -2 & -1 & -10 & 12 \\ 0 & 0 & -1 & 1 \\ -6 & -4 & -32 & 39 \end{pmatrix}.$$

Problem 2. Let $A = \begin{pmatrix} 1 & -2 & 4 & 1 \\ 2 & 3 & 2 & 0 \\ 2 & 0 & -1 & 1 \\ 2 & 0 & 0 & 1 \end{pmatrix}$.

(i) Evaluate the determinant of the matrix A.

Alternative solution: We have transformed A into the identity matrix using elementary row operations. These included no row exchanges and three row multiplications, each time by -1.

It follows that $\det I = (-1)^3 \det A$.
$\implies \det A = -\det I = -1$.

Problem 3. Determine which of the following subsets of $\mathbb{R}^3$ are subspaces. Briefly explain.

A subset of $\mathbb{R}^3$ is a subspace if it is closed under addition and scalar multiplication. Besides, the subset must not be empty.

(i) The set S_1 of vectors $(x, y, z) \in \mathbb{R}^3$ such that $xyz = 0$.

$(0, 0, 0) \in S_1 \implies S_1$ is not empty.

$xyz = 0 \implies (rx)(ry)(rz) = r^3 xyz = 0.$
That is, $\mathbf{v} = (x, y, z) \in S_1 \implies r\mathbf{v} = (rx, ry, rz) \in S_1$.
Hence S_1 is closed under scalar multiplication.

However S_1 is not closed under addition.
Counterexample: $(1, 1, 0) + (0, 0, 1) = (1, 1, 1)$.

Problem 3. Determine which of the following subsets of $\mathbb{R}^3$ are subspaces. Briefly explain.

A subset of $\mathbb{R}^3$ is a subspace if it is closed under addition and scalar multiplication. Besides, the subset must not be empty.

(ii) The set S_2 of vectors $(x, y, z) \in \mathbb{R}^3$ such that $x + y + z = 0$.

$(0, 0, 0) \in S_2 \implies S_2$ is not empty.

$x + y + z = 0 \implies rx + ry + rz = r(x + y + z) = 0.$
Hence S_2 is closed under scalar multiplication.

$x + y + z = x' + y' + z' = 0 \implies$
$(x+x')+(y+y')+(z+z') = (x+y+z)+(x'+y'+z') = 0.$
That is, $\mathbf{v} = (x, y, z),\ \mathbf{v}' = (x', y', z') \in S_2$
$\implies \mathbf{v} + \mathbf{v}' = (x + x', y + y', z + z') \in S_2.$
Hence S_2 is closed under addition.

(iii) The set S_3 of vectors $(x, y, z) \in \mathbb{R}^3$ such that $y^2 + z^2 = 0$.

$y^2 + z^2 = 0 \iff y = z = 0.$

S_3 is a nonempty set closed under addition and scalar multiplication.

(iv) The set S_4 of vectors $(x, y, z) \in \mathbb{R}^3$ such that $y^2 - z^2 = 0$.

S_4 is a nonempty set closed under scalar multiplication.
However S_4 is not closed under addition.
Counterexample: $(0, 1, 1) + (0, 1, -1) = (0, 2, 0)$.

Problem 4. Let $B = \begin{pmatrix} 0 & -1 & 4 & 1 \\ 1 & 1 & 2 & -1 \\ -3 & 0 & -1 & 0 \\ 2 & -1 & 0 & 1 \end{pmatrix}$.

(i) Find the rank and the nullity of the matrix B.

The rank (= dimension of the row space) and the nullity (= dimension of the nullspace) of a matrix are preserved under elementary row operations. We apply such operations to convert the matrix B into row echelon form.

Interchange the 1st row with the 2nd row:

$$\to \begin{pmatrix} 1 & 1 & 2 & -1 \\ 0 & -1 & 4 & 1 \\ -3 & 0 & -1 & 0 \\ 2 & -1 & 0 & 1 \end{pmatrix}$$

Add 3 times the 1st row to the 3rd row, then subtract 2 times the 1st row from the 4th row:

$$\rightarrow \begin{pmatrix} 1 & 1 & 2 & -1 \\ 0 & -1 & 4 & 1 \\ 0 & 3 & 5 & -3 \\ 2 & -1 & 0 & 1 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 1 & 2 & -1 \\ 0 & -1 & 4 & 1 \\ 0 & 3 & 5 & -3 \\ 0 & -3 & -4 & 3 \end{pmatrix}$$

Multiply the 2nd row by -1:

$$\rightarrow \begin{pmatrix} 1 & 1 & 2 & -1 \\ 0 & 1 & -4 & -1 \\ 0 & 3 & 5 & -3 \\ 0 & -3 & -4 & 3 \end{pmatrix}$$

Add the 4th row to the 3rd row:

$$\rightarrow \begin{pmatrix} 1 & 1 & 2 & -1 \\ 0 & 1 & -4 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & -3 & -4 & 3 \end{pmatrix}$$

Add 3 times the 2nd row to the 4th row:

$$\rightarrow \begin{pmatrix} 1 & 1 & 2 & -1 \\ 0 & 1 & -4 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & -16 & 0 \end{pmatrix}$$

Add 16 times the 3rd row to the 4th row:

$$\rightarrow \begin{pmatrix} 1 & 1 & 2 & -1 \\ 0 & 1 & -4 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Now that the matrix is in row echelon form, its rank equals the number of nonzero rows, which is 3. Since

$$(\text{rank of } B) + (\text{nullity of } B) = (\text{the number of columns of } B) = 4,$$

it follows that the nullity of B equals 1.

Problem 4. Let $B = \begin{pmatrix} 0 & -1 & 4 & 1 \\ 1 & 1 & 2 & -1 \\ -3 & 0 & -1 & 0 \\ 2 & -1 & 0 & 1 \end{pmatrix}$.

(ii) Find a basis for the row space of B, then extend this basis to a basis for $\mathbb{R}^4$.

The row space of a matrix is invariant under elementary row operations. Therefore the row space of the matrix B is the same as the row space of its row echelon form:

$$\begin{pmatrix} 0 & -1 & 4 & 1 \\ 1 & 1 & 2 & -1 \\ -3 & 0 & -1 & 0 \\ 2 & -1 & 0 & 1 \end{pmatrix} \to \begin{pmatrix} 1 & 1 & 2 & -1 \\ 0 & 1 & -4 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

The nonzero rows of the latter matrix are linearly independent so that they form a basis for its row space:

$$\mathbf{v}_1 = (1, 1, 2, -1),\ \ \mathbf{v}_2 = (0, 1, -4, -1),\ \ \mathbf{v}_3 = (0, 0, 1, 0).$$

To extend the basis $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ to a basis for $\mathbb{R}^4$, we need one vector $\mathbf{v}_4 \in \mathbb{R}^4$ that is not a linear combination of $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$.

It is known that at least one of the vectors $\mathbf{e}_1 = (1, 0, 0, 0)$, $\mathbf{e}_2 = (0, 1, 0, 0)$, $\mathbf{e}_3 = (0, 0, 1, 0)$, and $\mathbf{e}_4 = (0, 0, 0, 1)$ can be chosen as $\mathbf{v}_4$ (since $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3, \mathbf{e}_4$ is a spanning set for $\mathbb{R}^4$).

In particular, the vectors $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{e}_4$ form a basis for $\mathbb{R}^4$. This follows from the fact that the 4×4 matrix whose rows are these vectors is not singular:

$$\begin{vmatrix} 1 & 1 & 2 & -1 \\ 0 & 1 & -4 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{vmatrix} = 1 \neq 0.$$

Problem 4. Let $B = \begin{pmatrix} 0 & -1 & 4 & 1 \\ 1 & 1 & 2 & -1 \\ -3 & 0 & -1 & 0 \\ 2 & -1 & 0 & 1 \end{pmatrix}$.

(iii) Find a basis for the nullspace of B.

The nullspace of B is the solution set of the system of linear homogeneous equations with B as the coefficient matrix. To solve the system, we convert B to reduced row echelon form:

$$\to \begin{pmatrix} 1 & 1 & 2 & -1 \\ 0 & 1 & -4 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\implies \quad x_1 = x_2 - x_4 = x_3 = 0$$

General solution: $(x_1, x_2, x_3, x_4) = (0, t, 0, t) = t(0, 1, 0, 1)$.

Thus the vector $(0, 1, 0, 1)$ forms a basis for the nullspace of B.

Bonus Problem 5. Show that the functions $f_1(x) = x$, $f_2(x) = xe^x$, and $f_3(x) = e^{-x}$ are linearly independent in the vector space $C^\infty(\mathbb{R})$.

The functions f_1, f_2, f_3 are linearly independent whenever the Wronskian $W[f_1, f_2, f_3]$ is not identically zero.

$$W[f_1, f_2, f_3](x) = \begin{vmatrix} f_1(x) & f_2(x) & f_3(x) \\ f_1'(x) & f_2'(x) & f_3'(x) \\ f_1''(x) & f_2''(x) & f_3''(x) \end{vmatrix} = \begin{vmatrix} x & xe^x & e^{-x} \\ 1 & e^x+xe^x & -e^{-x} \\ 0 & 2e^x+xe^x & e^{-x} \end{vmatrix}$$

$$= e^{-x} \begin{vmatrix} x & xe^x & 1 \\ 1 & e^x+xe^x & -1 \\ 0 & 2e^x+xe^x & 1 \end{vmatrix} = \begin{vmatrix} x & x & 1 \\ 1 & 1+x & -1 \\ 0 & 2+x & 1 \end{vmatrix}$$

$$= x \begin{vmatrix} 1+x & -1 \\ 2+x & 1 \end{vmatrix} - \begin{vmatrix} x & 1 \\ 2+x & 1 \end{vmatrix} = x(2x+3)+2 = 2x^2+3x+2.$$

The polynomial $2x^2 + 3x + 2$ is never zero.

Bonus Problem 5. Show that the functions $f_1(x) = x$, $f_2(x) = xe^x$, and $f_3(x) = e^{-x}$ are linearly independent in the vector space $C^\infty(\mathbb{R})$.

Alternative solution: Suppose that $af_1(x)+bf_2(x)+cf_3(x) = 0$ for all $x \in \mathbb{R}$, where a, b, c are constants. We have to show that $a = b = c = 0$.

Let us differentiate this identity:

$$ax + bxe^x + ce^{-x} = 0,$$
$$a + be^x + bxe^x - ce^{-x} = 0,$$
$$2be^x + bxe^x + ce^{-x} = 0,$$
$$3be^x + bxe^x - ce^{-x} = 0,$$
$$4be^x + bxe^x + ce^{-x} = 0.$$

(the 5th identity)−(the 3rd identity): $2be^x = 0 \implies b = 0.$

Substitute $b = 0$ in the 3rd identity: $ce^{-x} = 0 \implies c = 0.$

Substitute $b = c = 0$ in the 2nd identity: $a = 0.$

Bonus Problem 5. Show that the functions $f_1(x) = x$, $f_2(x) = xe^x$, and $f_3(x) = e^{-x}$ are linearly independent in the vector space $C^\infty(\mathbb{R})$.

Alternative solution: Suppose that $ax + bxe^x + ce^{-x} = 0$ for all $x \in \mathbb{R}$, where a, b, c are constants. We have to show that $a = b = c = 0$.

For any $x \neq 0$ divide both sides of the identity by xe^x:

$$ae^{-x} + b + cx^{-1}e^{-2x} = 0.$$

The left-hand side approaches b as $x \to +\infty$. $\implies b = 0$

Now $ax + ce^{-x} = 0$ for all $x \in \mathbb{R}$. For any $x \neq 0$ divide both sides of the identity by x:

$$a + cx^{-1}e^{-x} = 0.$$

The left-hand side approaches a as $x \to +\infty$. $\implies a = 0$

Now $ce^{-x} = 0 \implies c = 0$.

Bonus Problem 6. Let V be a finite-dimensional vector space and V_0 be a proper subspace of V (where proper means that $V_0 \neq V$). Prove that $\dim V_0 < \dim V$.

Any vector space has a basis. Let $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k$ be a basis for V_0.

Vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k$ are linearly independent in V since they are linearly independent in V_0. Therefore we can extend this collection of vectors to a basis for V by adding some vectors $\mathbf{w}_1, \ldots, \mathbf{w}_m$. As $V_0 \neq V$, we do need to add some vectors, i.e., $m \geq 1$.

Thus $\dim V_0 = k$ and $\dim V = k + m > k$.