MATH 323

Linear Algebra

Lecture 22:
Eigenvalues and eigenvectors.
Characteristic polynomial.

Eigenvalues and eigenvectors of a matrix

Definition. Let A be an $n\times n$ matrix. A number $\lambda \in \mathbb{R}$ is called an **eigenvalue** of the matrix A if $\boxed{A\mathbf{v} = \lambda\mathbf{v}}$ for a nonzero column vector $\mathbf{v} \in \mathbb{R}^n$.

The vector $\mathbf{v}$ is called an **eigenvector** of A belonging to (or associated with) the eigenvalue λ.

Remarks. • Alternative notation:
eigenvalue = **characteristic value**,
eigenvector = **characteristic vector**.

• The zero vector is never considered an eigenvector.

Example. $A = \begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix}$.

$$\begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 2 \\ 0 \end{pmatrix} = 2 \begin{pmatrix} 1 \\ 0 \end{pmatrix},$$

$$\begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix} \begin{pmatrix} 0 \\ -2 \end{pmatrix} = \begin{pmatrix} 0 \\ -6 \end{pmatrix} = 3 \begin{pmatrix} 0 \\ -2 \end{pmatrix}.$$

Hence $(1, 0)^T$ is an eigenvector of A belonging to the eigenvalue 2, while $(0, -2)^T$ is an eigenvector of A belonging to the eigenvalue 3.

Example. $A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$.

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \quad \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ -1 \end{pmatrix} = \begin{pmatrix} -1 \\ 1 \end{pmatrix}.$$

Hence $(1,1)^T$ is an eigenvector of A belonging to the eigenvalue 1, while $(1,-1)^T$ is an eigenvector of A belonging to the eigenvalue -1.

Vectors $\mathbf{v}_1 = (1,1)$ and $\mathbf{v}_2 = (1,-1)$ form a basis for $\mathbb{R}^2$. Consider a linear operator $L : \mathbb{R}^2 \to \mathbb{R}^2$ given by $L(\mathbf{x}) = A\mathbf{x}$. The matrix of L with respect to the basis $\mathbf{v}_1, \mathbf{v}_2$ is $B = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$.

Let A be an $n \times n$ matrix. Consider a linear operator $L : \mathbb{R}^n \to \mathbb{R}^n$ given by $L(\mathbf{x}) = A\mathbf{x}$.

Let $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n$ be a nonstandard basis for $\mathbb{R}^n$ and B be the matrix of the operator L with respect to this basis.

Theorem The matrix B is diagonal if and only if vectors $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n$ are eigenvectors of A.

If this is the case, then the diagonal entries of the matrix B are the corresponding eigenvalues of A.

$$A\mathbf{v}_i = \lambda_i \mathbf{v}_i \iff B = \begin{pmatrix} \lambda_1 & & & O \\ & \lambda_2 & & \\ & & \ddots & \\ O & & & \lambda_n \end{pmatrix}$$

Eigenspaces

Let A be an $n \times n$ matrix. Let $\mathbf{v}$ be an eigenvector of A belonging to an eigenvalue λ.

Then $A\mathbf{v} = \lambda\mathbf{v} \implies A\mathbf{v} = (\lambda I)\mathbf{v} \implies (A - \lambda I)\mathbf{v} = \mathbf{0}$.

Hence $\mathbf{v} \in N(A - \lambda I)$, the nullspace of the matrix $A - \lambda I$.

Conversely, if $\mathbf{x} \in N(A - \lambda I)$ then $A\mathbf{x} = \lambda\mathbf{x}$. Thus the eigenvectors of A belonging to the eigenvalue λ are nonzero vectors from $N(A - \lambda I)$.

Definition. If $N(A - \lambda I) \neq \{\mathbf{0}\}$ then it is called the **eigenspace** of the matrix A corresponding to the eigenvalue λ.

How to find eigenvalues and eigenvectors?

Theorem Given a square matrix A and a scalar λ, the following statements are equivalent:

- λ is an eigenvalue of A,
- $N(A - \lambda I) \neq \{\mathbf{0}\}$,
- the matrix $A - \lambda I$ is singular,
- $\det(A - \lambda I) = 0$.

Definition. $\det(A - \lambda I) = 0$ is called the **characteristic equation** of the matrix A.

Eigenvalues λ of A are roots of the characteristic equation. Associated eigenvectors of A are nonzero solutions of the equation $(A - \lambda I)\mathbf{x} = \mathbf{0}$.

Example. $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$.

$$\det(A - \lambda I) = \begin{vmatrix} a - \lambda & b \\ c & d - \lambda \end{vmatrix}$$

$$= (a - \lambda)(d - \lambda) - bc$$

$$= \lambda^2 - (a + d)\lambda + (ad - bc).$$

Example. $A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$.

$$\det(A - \lambda I) = \begin{vmatrix} a_{11} - \lambda & a_{12} & a_{13} \\ a_{21} & a_{22} - \lambda & a_{23} \\ a_{31} & a_{32} & a_{33} - \lambda \end{vmatrix}$$

$$= -\lambda^3 + c_1\lambda^2 - c_2\lambda + c_3,$$

where $c_1 = a_{11} + a_{22} + a_{33}$ (the *trace* of A),

$c_2 = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} + \begin{vmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{vmatrix} + \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix}$,

$c_3 = \det A$.

Theorem. Let $A = (a_{ij})$ be an $n\times n$ matrix. Then $\det(A - \lambda I)$ is a polynomial of λ of degree n:

$$\det(A - \lambda I) = (-1)^n \lambda^n + c_1 \lambda^{n-1} + \cdots + c_{n-1}\lambda + c_n.$$

Furthermore, $(-1)^{n-1} c_1 = a_{11} + a_{22} + \cdots + a_{nn}$ and $c_n = \det A$.

Definition. The polynomial $p(\lambda) = \det(A - \lambda I)$ is called the **characteristic polynomial** of the matrix A.

Corollary Any $n\times n$ matrix has at most n eigenvalues.

Example. $A = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$.

Characteristic equation: $\begin{vmatrix} 2-\lambda & 1 \\ 1 & 2-\lambda \end{vmatrix} = 0.$

$(2-\lambda)^2 - 1 = 0 \implies \lambda_1 = 1, \;\; \lambda_2 = 3.$

$$(A - I)\mathbf{x} = \mathbf{0} \iff \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\iff \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \iff x + y = 0.$$

The general solution is $(-t, t) = t(-1, 1), \;\; t \in \mathbb{R}$.
Thus $\mathbf{v}_1 = (-1, 1)$ is an eigenvector associated with the eigenvalue 1. The corresponding eigenspace is the line spanned by $\mathbf{v}_1$.

$$(A-3I)\mathbf{x}=\mathbf{0} \iff \begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}=\begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\iff \begin{pmatrix} 1 & -1 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}=\begin{pmatrix} 0 \\ 0 \end{pmatrix} \iff x-y=0.$$

The general solution is $(t,t)=t(1,1),\ \ t\in\mathbb{R}$. Thus $\mathbf{v}_2=(1,1)$ is an eigenvector associated with the eigenvalue 3. The corresponding eigenspace is the line spanned by $\mathbf{v}_2$.

Summary. $A = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$.

- The matrix A has two eigenvalues: 1 and 3.

- The eigenspace of A associated with the eigenvalue 1 is the line $t(-1, 1)$.

- The eigenspace of A associated with the eigenvalue 3 is the line $t(1, 1)$.

- Eigenvectors $\mathbf{v}_1 = (-1, 1)$ and $\mathbf{v}_2 = (1, 1)$ of the matrix A form an orthogonal basis for $\mathbb{R}^2$.

- Geometrically, the mapping $\mathbf{x} \mapsto A\mathbf{x}$ is a stretch by a factor of 3 away from the line $x + y = 0$ in the orthogonal direction.

Example. $A = \begin{pmatrix} 1 & 1 & -1 \\ 1 & 1 & 1 \\ 0 & 0 & 2 \end{pmatrix}$.

Characteristic equation:

$$\begin{vmatrix} 1-\lambda & 1 & -1 \\ 1 & 1-\lambda & 1 \\ 0 & 0 & 2-\lambda \end{vmatrix} = 0.$$

Expand the determinant by the 3rd row:

$$(2-\lambda) \begin{vmatrix} 1-\lambda & 1 \\ 1 & 1-\lambda \end{vmatrix} = 0.$$

$$\big((1-\lambda)^2 - 1\big)(2-\lambda) = 0 \iff -\lambda(2-\lambda)^2 = 0$$

$$\implies \quad \lambda_1 = 0, \;\; \lambda_2 = 2.$$

$$A\mathbf{x} = \mathbf{0} \iff \begin{pmatrix} 1 & 1 & -1 \\ 1 & 1 & 1 \\ 0 & 0 & 2 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

Convert the matrix to reduced row echelon form:

$$\begin{pmatrix} 1 & 1 & -1 \\ 1 & 1 & 1 \\ 0 & 0 & 2 \end{pmatrix} \to \begin{pmatrix} 1 & 1 & -1 \\ 0 & 0 & 2 \\ 0 & 0 & 2 \end{pmatrix} \to \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

$$A\mathbf{x} = \mathbf{0} \iff \begin{cases} x + y = 0, \\ z = 0. \end{cases}$$

The general solution is $(-t, t, 0) = t(-1, 1, 0)$, $t \in \mathbb{R}$. Thus $\mathbf{v}_1 = (-1, 1, 0)$ is an eigenvector associated with the eigenvalue 0. The corresponding eigenspace is the line spanned by $\mathbf{v}_1$.

$$(A-2I)\mathbf{x}=\mathbf{0} \iff \begin{pmatrix} -1 & 1 & -1 \\ 1 & -1 & 1 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\iff \begin{pmatrix} 1 & -1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \iff x-y+z=0.$$

The general solution is $x=t-s$, $y=t$, $z=s$, where $t,s\in\mathbb{R}$. Equivalently,

$$\mathbf{x}=(t-s,t,s)=t(1,1,0)+s(-1,0,1).$$

Thus $\mathbf{v}_2=(1,1,0)$ and $\mathbf{v}_3=(-1,0,1)$ are eigenvectors associated with the eigenvalue 2. The corresponding eigenspace is the plane spanned by $\mathbf{v}_2$ and $\mathbf{v}_3$.

Summary. $A = \begin{pmatrix} 1 & 1 & -1 \\ 1 & 1 & 1 \\ 0 & 0 & 2 \end{pmatrix}$.

- The matrix A has two eigenvalues: 0 and 2.
- The eigenvalue 0 is *simple:* the corresponding eigenspace is a line.
- The eigenvalue 2 is of *multiplicity* 2: the corresponding eigenspace is a plane.
- Eigenvectors $\mathbf{v}_1 = (-1, 1, 0)$, $\mathbf{v}_2 = (1, 1, 0)$, and $\mathbf{v}_3 = (-1, 0, 1)$ of the matrix A form a basis for $\mathbb{R}^3$.
- Geometrically, the map $\mathbf{x} \mapsto A\mathbf{x}$ is the projection on the plane $\mathrm{Span}(\mathbf{v}_2, \mathbf{v}_3)$ along the lines parallel to $\mathbf{v}_1$ with the subsequent scaling by a factor of 2.

Eigenvalues and eigenvectors of an operator

Definition. Let V be a vector space and $L : V \to V$ be a linear operator. A number λ is called an **eigenvalue** of the operator L if $\boxed{L(\mathbf{v}) = \lambda\mathbf{v}}$ for a nonzero vector $\mathbf{v} \in V$. The vector $\mathbf{v}$ is called an **eigenvector** of L associated with the eigenvalue λ.

(If V is a functional space then eigenvectors are also called **eigenfunctions**.)

If $V = \mathbb{R}^n$ then the linear operator L is given by $L(\mathbf{x}) = A\mathbf{x}$, where A is an $n \times n$ matrix.
In this case, eigenvalues and eigenvectors of the operator L are precisely eigenvalues and eigenvectors of the matrix A.

Suppose $L : V \to V$ is a linear operator on a **finite-dimensional** vector space V.

Let $\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_n$ be a basis for V and $g : V \to \mathbb{R}^n$ be the corresponding coordinate mapping. Let A be the matrix of L with respect to this basis. Then

$$L(\mathbf{v}) = \lambda\mathbf{v} \iff A\,g(\mathbf{v}) = \lambda\,g(\mathbf{v}).$$

Hence the eigenvalues of L coincide with those of the matrix A. Moreover, the associated eigenvectors of A are coordinates of the eigenvectors of L.

Definition. The characteristic polynomial $p(\lambda) = \det(A - \lambda I)$ of the matrix A is called the **characteristic polynomial** of the operator L.

Then eigenvalues of L are roots of its characteristic polynomial.

Theorem. The characteristic polynomial of the operator L is well defined. That is, it does not depend on the choice of a basis.

Proof: Let B be the matrix of L with respect to a different basis $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n$. Then $A = UBU^{-1}$, where U is the transition matrix from the basis $\mathbf{v}_1, \dots, \mathbf{v}_n$ to $\mathbf{u}_1, \dots, \mathbf{u}_n$. We have to show that $\det(A - \lambda I) = \det(B - \lambda I)$ for all $\lambda \in \mathbb{R}$. We obtain

$$\det(A - \lambda I) = \det(UBU^{-1} - \lambda I)$$
$$= \det\bigl(UBU^{-1} - U(\lambda I)U^{-1}\bigr) = \det\bigl(U(B - \lambda I)U^{-1}\bigr)$$
$$= \det(U)\,\det(B - \lambda I)\,\det(U^{-1}) = \det(B - \lambda I).$$

Eigenspaces

Let $L : V \to V$ be a linear operator.

For any $\lambda \in \mathbb{R}$, let V_λ denotes the set of all solutions of the equation $L(\mathbf{x}) = \lambda\mathbf{x}$.

Then V_λ is a *subspace* of V since V_λ is the *kernel* of a linear operator given by $\mathbf{x} \mapsto L(\mathbf{x}) - \lambda\mathbf{x}$.

V_λ minus the zero vector is the set of all eigenvectors of L associated with the eigenvalue λ. In particular, $\lambda \in \mathbb{R}$ is an eigenvalue of L if and only if $V_\lambda \neq \{\mathbf{0}\}$.

If $V_\lambda \neq \{\mathbf{0}\}$ then it is called the **eigenspace** of L corresponding to the eigenvalue λ.

Example. $V = C^\infty(\mathbb{R}),\ D : V \to V,\ Df = f'.$

A function $f \in C^\infty(\mathbb{R})$ is an eigenfunction of the operator D belonging to an eigenvalue λ if $f'(x) = \lambda f(x)$ for all $x \in \mathbb{R}$.

It follows that $f(x) = ce^{\lambda x}$, where c is a nonzero constant.

Thus each $\lambda \in \mathbb{R}$ is an eigenvalue of D.
The corresponding eigenspace is spanned by $e^{\lambda x}$.

Theorem If $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_k$ are eigenvectors of a linear operator L associated with distinct eigenvalues $\lambda_1, \lambda_2, \dots, \lambda_k$, then $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_k$ are linearly independent.

Proof: The proof is by induction on k. The case $k = 1$ is trivial. Now assume $k > 1$ and the theorem holds for $k - 1$ vectors. Suppose that $t_1\mathbf{v}_1 + t_2\mathbf{v}_2 + \cdots + t_k\mathbf{v}_k = \mathbf{0}$ for some $t_i \in \mathbb{R}$. We have to show that $t_i = 0$, $1 \leq i \leq k$.

$$L(t_1\mathbf{v}_1 + t_2\mathbf{v}_2 + \cdots + t_k\mathbf{v}_k) = L(\mathbf{0}),$$
$$t_1L(\mathbf{v}_1) + t_2L(\mathbf{v}_2) + \cdots + t_kL(\mathbf{v}_k) = \mathbf{0},$$
$$t_1\lambda_1\mathbf{v}_1 + t_2\lambda_2\mathbf{v}_2 + \cdots + t_k\lambda_k\mathbf{v}_k = \mathbf{0}.$$

It follows that

$$t_1\lambda_1\mathbf{v}_1 + t_2\lambda_2\mathbf{v}_2 + \cdots + t_k\lambda_k\mathbf{v}_k - \lambda_1(t_1\mathbf{v}_1 + t_2\mathbf{v}_2 + \cdots + t_k\mathbf{v}_k) = \mathbf{0}$$
$$\implies t_2(\lambda_2 - \lambda_1)\mathbf{v}_2 + \cdots + t_k(\lambda_k - \lambda_1)\mathbf{v}_k = \mathbf{0}.$$

By the inductive assumption, vectors $\mathbf{v}_2, \dots, \mathbf{v}_k$ are linearly independent. Hence $t_2(\lambda_2 - \lambda_1) = \cdots = t_k(\lambda_k - \lambda_1) = 0$ $\implies t_2 = \cdots = t_k = 0$. Then $t_1\mathbf{v}_1 = \mathbf{0}$ so that $t_1 = 0$ too.

Corollary 1 If $\lambda_1, \lambda_2, \ldots, \lambda_k$ are distinct real numbers, then the functions $e^{\lambda_1 x}, e^{\lambda_2 x}, \ldots, e^{\lambda_k x}$ are linearly independent.

Proof: Consider a linear operator $D : C^\infty(\mathbb{R}) \to C^\infty(\mathbb{R})$ given by $Df = f'$. Then $e^{\lambda_1 x}, \ldots, e^{\lambda_k x}$ are eigenfunctions of D associated with distinct eigenvalues $\lambda_1, \ldots, \lambda_k$. By the theorem, the eigenfunctions are linearly independent.

Corollary 2 If $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k$ are eigenvectors of a matrix A associated with distinct eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_k$, then $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k$ are linearly independent.

Corollary 3 Let A be an $n \times n$ matrix such that the characteristic equation $\det(A - \lambda I) = 0$ has n distinct real roots. Then $\mathbb{R}^n$ has a basis consisting of eigenvectors of A.

Proof: Let $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$ be eigenvectors of the matrix A belonging to n distinct real roots of its characteristic equation. By Corollary 2, these vectors are linearly independent. Therefore they form a basis for $\mathbb{R}^n$.