MATH 323

Linear Algebra

Lecture 26: Orthogonal polynomials. Review for the final exam.

Orthogonal polynomials

$\mathcal{P}$: the vector space of all polynomials with real coefficients: $p(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n$.

Basis for $\mathcal{P}$: $1, x, x^2, \dots, x^n, \dots$

Suppose that $\mathcal{P}$ is endowed with an inner product.

Definition. **Orthogonal polynomials** (relative to the inner product) are polynomials $p_0, p_1, p_2, \dots$ such that $\deg p_n = n$ (p_0 is a nonzero constant) and $\langle p_n, p_m \rangle = 0$ for $n \neq m$.

Orthogonal polynomials can be obtained by applying the *Gram-Schmidt orthogonalization process* to the basis $1, x, x^2, \ldots$:

$$p_0(x) = 1,$$

$$p_1(x) = x - \frac{\langle x, p_0 \rangle}{\langle p_0, p_0 \rangle} p_0(x),$$

$$p_2(x) = x^2 - \frac{\langle x^2, p_0 \rangle}{\langle p_0, p_0 \rangle} p_0(x) - \frac{\langle x^2, p_1 \rangle}{\langle p_1, p_1 \rangle} p_1(x),$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$p_n(x) = x^n - \frac{\langle x^n, p_0 \rangle}{\langle p_0, p_0 \rangle} p_0(x) - \cdots - \frac{\langle x^n, p_{n-1} \rangle}{\langle p_{n-1}, p_{n-1} \rangle} p_{n-1}(x),$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

Then $p_0, p_1, p_2, \ldots$ are orthogonal polynomials.

Theorem (a) Orthogonal polynomials always exist.

(b) The orthogonal polynomial of a fixed degree is unique up to scaling.

(c) A polynomial $p \neq 0$ is an orthogonal polynomial if and only if $\langle p, q \rangle = 0$ for any polynomial q with $\deg q < \deg p$.

(d) A polynomial $p \neq 0$ is an orthogonal polynomial if and only if $\langle p, x^k \rangle = 0$ for any $0 \leq k < \deg p$.

Proof of statement (b): Suppose that P and R are two orthogonal polynomials of the same degree n. Then $P(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0$ and $R(x) = b_n x^n + b_{n-1} x^{n-1} + \cdots + b_1 x + b_0$, where $a_n, b_n \neq 0$. Consider a polynomial $Q(x) = b_n P(x) - a_n R(x)$. By construction, $\deg Q < n$. It follows from statement (c) that $\langle P, Q \rangle = \langle R, Q \rangle = 0$. Then

$$\langle Q, Q \rangle = \langle b_n P - a_n R, Q \rangle = b_n \langle P, Q \rangle - a_n \langle R, Q \rangle = 0,$$

which means that $Q = 0$. Thus $R(x) = (a_n^{-1} b_n)\, P(x)$.

Example. $\langle p, q \rangle = \int_{-1}^{1} p(x)q(x)\,dx.$

Note that $\langle x^m, x^n \rangle = \int_{-1}^{1} x^{m+n}\,dx = 0$ if $m+n$ is odd. Hence $p_{2k}(x)$ contains only even powers of x while $p_{2k+1}(x)$ contains only odd powers of x.

$p_0(x) = 1,$
$p_1(x) = x,$

$$p_2(x) = x^2 - \frac{\langle x^2, 1 \rangle}{\langle 1, 1 \rangle} = x^2 - \frac{1}{3},$$

$$p_3(x) = x^3 - \frac{\langle x^3, x \rangle}{\langle x, x \rangle} x = x^3 - \frac{3}{5}x.$$

$p_0, p_1, p_2, \dots$ are called the **Legendre polynomials**.

Instead of normalization, the orthogonal polynomials are subject to **standardization**.

The standardization for the Legendre polynomials is $P_n(1) = 1$. In particular, $P_0(x) = 1$, $P_1(x) = x$, $P_2(x) = \frac{1}{2}(3x^2 - 1)$, $P_3(x) = \frac{1}{2}(5x^3 - 3x)$.

Problem. Find $P_4(x)$.

Let $P_4(x) = a_4x^4 + a_3x^3 + a_2x^2 + a_1x + a_0$.
We know that $P_4(1) = 1$ and $\langle P_4, x^k \rangle = 0$ for $0 \le k \le 3$.

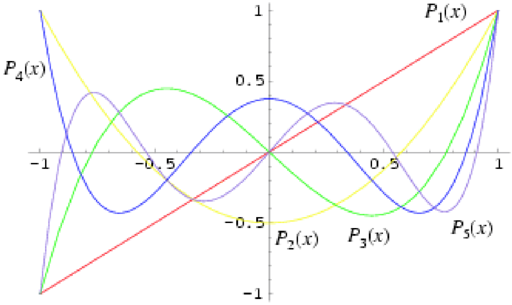
$P_4(1) = a_4 + a_3 + a_2 + a_1 + a_0$,
$\langle P_4, 1 \rangle = \frac{2}{5}a_4 + \frac{2}{3}a_2 + 2a_0$, $\langle P_4, x \rangle = \frac{2}{5}a_3 + \frac{2}{3}a_1$,
$\langle P_4, x^2 \rangle = \frac{2}{7}a_4 + \frac{2}{5}a_2 + \frac{2}{3}a_0$, $\langle P_4, x^3 \rangle = \frac{2}{7}a_3 + \frac{2}{5}a_1$.

$$\begin{cases} a_4 + a_3 + a_2 + a_1 + a_0 = 1 \\ \frac{2}{5}a_4 + \frac{2}{3}a_2 + 2a_0 = 0 \\ \frac{2}{5}a_3 + \frac{2}{3}a_1 = 0 \\ \frac{2}{7}a_4 + \frac{2}{5}a_2 + \frac{2}{3}a_0 = 0 \\ \frac{2}{7}a_3 + \frac{2}{5}a_1 = 0 \end{cases}$$

$$\begin{cases} \frac{2}{5}a_3 + \frac{2}{3}a_1 = 0 \\ \frac{2}{7}a_3 + \frac{2}{5}a_1 = 0 \end{cases} \implies a_1 = a_3 = 0$$

$$\begin{cases} a_4 + a_2 + a_0 = 1 \\ \frac{2}{5}a_4 + \frac{2}{3}a_2 + 2a_0 = 0 \\ \frac{2}{7}a_4 + \frac{2}{5}a_2 + \frac{2}{3}a_0 = 0 \end{cases} \iff \begin{cases} a_4 = \frac{35}{8} \\ a_2 = -\frac{30}{8} \\ a_0 = \frac{3}{8} \end{cases}$$

Thus $P_4(x) = \frac{1}{8}(35x^4 - 30x^2 + 3)$.



Legendre polynomials

Problem. Find a quadratic polynomial that is the best least squares fit to the function $f(x) = |x|$ on the interval $[-1, 1]$.

The best least squares fit is a polynomial $p(x)$ that minimizes the distance relative to the integral norm

$$\|f - p\| = \left(\int_{-1}^{1} |f(x) - p(x)|^2 \, dx \right)^{1/2}$$

over all polynomials of degree 2.

The norm $\|f - p\|$ is minimal if p is the orthogonal projection of the function f on the subspace $\mathcal{P}_3$ of polynomials of degree at most 2.

The Legendre polynomials P_0, P_1, P_2 form an orthogonal basis for $\mathcal{P}_3$. Therefore

$$p(x) = \frac{\langle f, P_0\rangle}{\langle P_0, P_0\rangle} P_0(x) + \frac{\langle f, P_1\rangle}{\langle P_1, P_1\rangle} P_1(x) + \frac{\langle f, P_2\rangle}{\langle P_2, P_2\rangle} P_2(x).$$

$$\langle f, P_0\rangle = \int_{-1}^{1} |x|\, dx = 2\int_0^1 x\, dx = 1,$$

$$\langle f, P_1\rangle = \int_{-1}^{1} |x|\, x\, dx = 0,$$

$$\langle f, P_2\rangle = \int_{-1}^{1} |x|\, \frac{3x^2-1}{2}\, dx = \int_0^1 x(3x^2-1)\, dx = \frac{1}{4},$$

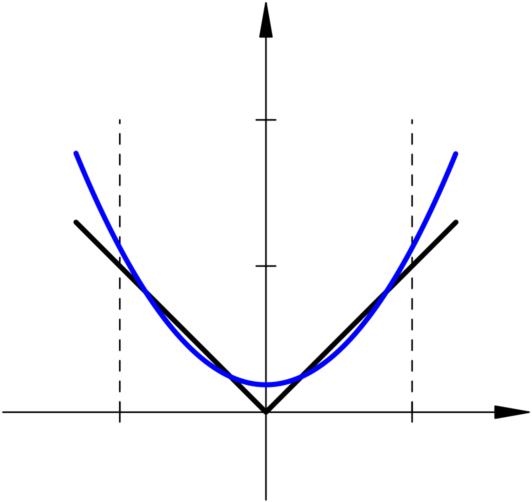
$$\langle P_0, P_0\rangle = \int_{-1}^{1} dx = 2, \qquad \langle P_2, P_2\rangle = \int_{-1}^{1} \left(\frac{3x^2-1}{2}\right)^2 dx = \frac{2}{5}.$$

In general, $\langle P_n, P_n\rangle = \dfrac{2}{2n+1}$.

Problem. Find a quadratic polynomial that is the best least squares fit to the function $f(x) = |x|$ on the interval $[-1, 1]$.

Solution: $p(x) = \frac{1}{2}P_0(x) + \frac{5}{8}P_2(x)$

$$= \frac{1}{2} + \frac{5}{16}(3x^2 - 1) = \frac{3}{16}(5x^2 + 1).$$



How to evaluate orthogonal polynomials

Suppose $p_0, p_1, p_2, \ldots$ are orthogonal polynomials with respect to an inner product of the form

$$\langle p, q \rangle = \int_a^b p(x)q(x)w(x)\,dx.$$

Theorem The polynomials satisfy recurrences

$$p_n(x) = (\alpha_n x + \beta_n)\,p_{n-1}(x) + \gamma_n\,p_{n-2}(x)$$

for all $n \geq 2$, where $\alpha_n, \beta_n, \gamma_n$ are some constants.

Recurrent formulas for the Legendre polynomials:

$$(n+1)P_{n+1}(x) = (2n+1)xP_n(x) - nP_{n-1}(x).$$

For example, $4P_4(x) = 7xP_3(x) - 3P_2(x)$.

Definition. **Chebyshev polynomials** $T_0, T_1, T_2, \ldots$ are orthogonal polynomials relative to the inner product

$$\langle p, q \rangle = \int_{-1}^{1} \frac{p(x)q(x)}{\sqrt{1-x^2}}\, dx,$$

with the standardization $T_n(1) = 1$.

Remark. "T" is like in "Tschebyscheff".

Change of variable in the integral: $x = \cos\phi$.

$$\langle p, q \rangle = -\int_0^{\pi} \frac{p(\cos\phi)\, q(\cos\phi)}{\sqrt{1-\cos^2\phi}} \cos'\phi\, d\phi$$

$$= \int_0^{\pi} p(\cos\phi)\, q(\cos\phi)\, d\phi.$$

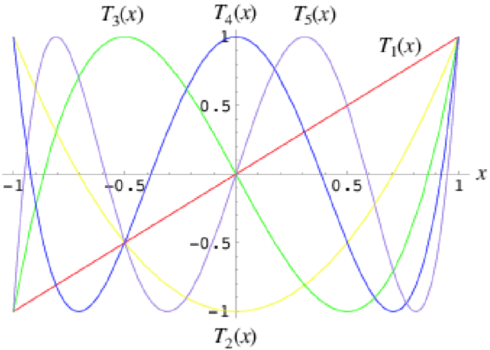
Theorem. $T_n(\cos\phi) = \cos n\phi$.

$$\langle T_n, T_m \rangle = \int_0^\pi T_n(\cos\phi)\, T_m(\cos\phi)\, d\phi$$
$$= \int_0^\pi \cos(n\phi)\cos(m\phi)\, d\phi = 0 \quad \text{if} \quad n \neq m.$$

Recurrent formula: $T_{n+1}(x) = 2xT_n(x) - T_{n-1}(x)$.

$T_0(x) = 1, \quad T_1(x) = x,$
$T_2(x) = 2x^2 - 1,$
$T_3(x) = 4x^3 - 3x,$
$T_4(x) = 8x^4 - 8x^2 + 1, \ \ldots$

That is, $\cos 2\phi = 2\cos^2\phi - 1,$
$\cos 3\phi = 4\cos^3\phi - 3\cos\phi,$
$\cos 4\phi = 8\cos^4\phi - 8\cos^2\phi + 1, \ \ldots$



Chebyshev polynomials

Topics for the final exam: Part I

Elementary linear algebra (Leon 1.1–1.5, 2.1–2.2)

- Systems of linear equations: elementary operations, Gaussian elimination, back substitution.

- Matrix of coefficients and augmented matrix. Elementary row operations, row echelon form and reduced row echelon form.

- Matrix algebra. Inverse matrix.

- Determinants: explicit formulas for 2×2 and 3×3 matrices, row and column expansions, elementary row and column operations.

Topics for the final exam: Part II

Abstract linear algebra (Leon 3.1–3.6, 4.1–4.3)

- Vector spaces (vectors, matrices, polynomials, functional spaces).
- Subspaces. Nullspace, column space, and row space of a matrix.
- Span, spanning set. Linear independence.
- Bases and dimension.
- Rank and nullity of a matrix.
- Coordinates relative to a basis.
- Change of basis, transition matrix.

- Linear transformations.
- Matrix transformations.
- Matrix of a linear transformation.
- Change of basis for a linear operator.
- Similarity of matrices.

Topics for the final exam: Parts III–IV

Advanced linear algebra (Leon 5.1–5.7, 6.1–6.3)

- Euclidean structure in $\mathbb{R}^n$ (length, angle, dot product).
- Orthogonal complement, orthogonal projection.
- Inner products and norms.
- Least squares problems.
- The Gram-Schmidt orthogonalization process.
- Orthogonal polynomials.

- Eigenvalues, eigenvectors, eigenspaces.
- Characteristic polynomial.
- Bases of eigenvectors, diagonalization.
- Matrix exponentials.
- Complex eigenvalues and eigenvectors.
- Orthogonal matrices.
- Rigid motions, rotations in space.

Proofs to know

Theorem 1 If two $n \times n$ matrices A and B are invertible, then the product AB is also invertible and $(AB)^{-1} = B^{-1}A^{-1}$.

Theorem 2 If an $n \times n$ matrix A is invertible, then for any n-dimensional column vector $\mathbf{b}$ the matrix equation $A\mathbf{x} = \mathbf{b}$ has a unique solution, which is $\mathbf{x} = A^{-1}\mathbf{b}$.

Proofs to know

Theorem 3 In any vector space, the zero vector is unique and the negative vector is unique.

Theorem 4 For any vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k$ in a vector space V, the set of all linear combinations $r_1\mathbf{v}_1 + r_2\mathbf{v}_2 + \cdots + r_k\mathbf{v}_k$, $r_i \in \mathbb{R}$ is a subspace of V.

Theorem 5 Vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k$ $(k \geq 2)$ are linearly dependent if and only if one of them is a linear combination of the other $k-1$ vectors.

Theorem 6 Functions $f_1, f_2, \ldots, f_n \in C[a, b]$ are linearly independent whenever their Wronskian $W[f_1, f_2, \ldots, f_n]$ is well defined and not identically zero on $[a, b]$.

Proofs to know

Theorem 7 Any linear mapping $L : \mathbb{R}^m \to \mathbb{R}^n$ is a matrix transformation.

Theorem 8 The orthogonal complement of the row space of a matrix A is the nullspace of A.

Theorem 9 If nonzero vectors in an inner product space are orthogonal to each other, then they are linearly independent.

Theorem 10 $\lambda \in \mathbb{R}$ is an eigenvalue of a matrix A if and only if $\det(A - \lambda I) = 0$.

Theorem 11 Similar matrices have the same characteristic polynomial.

Theorem 12 If $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_k$ are eigenvectors of the same linear operator L associated with distinct eigenvalues $\lambda_1, \lambda_2, \dots, \lambda_k$, then $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_k$ are linearly independent.

Proofs to know

Theorem 13 A square matrix is orthogonal if and only if its columns form an orthonormal set.

Problem. Consider a linear operator $L : \mathbb{R}^3 \to \mathbb{R}^3$ defined by $L(\mathbf{v}) = \mathbf{v}_0 \times \mathbf{v}$, where $\mathbf{v}_0 = (3/5, 0, -4/5)$.

(a) Find the matrix B of the operator L.
(b) Find the range and kernel of L.
(c) Find the eigenvalues of L.
(d) Find the matrix of the operator L^{2013} (L applied 2013 times).

$L(\mathbf{v}) = \mathbf{v}_0 \times \mathbf{v}, \;\; \mathbf{v}_0 = (3/5, 0, -4/5).$

Let $\mathbf{v} = (x, y, z) = x\mathbf{e}_1 + y\mathbf{e}_2 + z\mathbf{e}_3$. Then

$$L(\mathbf{v}) = \mathbf{v}_0 \times \mathbf{v} = \begin{vmatrix} \mathbf{e}_1 & \mathbf{e}_2 & \mathbf{e}_3 \\ 3/5 & 0 & -4/5 \\ x & y & z \end{vmatrix}$$

$$= \tfrac{4}{5}y\mathbf{e}_1 - \left(\tfrac{4}{5}x + \tfrac{3}{5}z\right)\mathbf{e}_2 + \tfrac{3}{5}y\mathbf{e}_3.$$

In particular, $L(\mathbf{e}_1) = -\frac{4}{5}\mathbf{e}_2$, $L(\mathbf{e}_2) = \frac{4}{5}\mathbf{e}_1 + \frac{3}{5}\mathbf{e}_3$, $L(\mathbf{e}_3) = -\frac{3}{5}\mathbf{e}_2$.

Therefore $B = \begin{pmatrix} 0 & 4/5 & 0 \\ -4/5 & 0 & -3/5 \\ 0 & 3/5 & 0 \end{pmatrix}$.

$$B = \begin{pmatrix} 0 & 4/5 & 0 \\ -4/5 & 0 & -3/5 \\ 0 & 3/5 & 0 \end{pmatrix}.$$

The range of the operator L is spanned by columns of the matrix B. It follows that $\mathrm{Range}(L)$ is the plane spanned by $\mathbf{v}_1 = (0, 1, 0)$ and $\mathbf{v}_2 = (4, 0, 3)$.

The kernel of L is the nullspace of the matrix B, i.e., the solution set for the equation $B\mathbf{x} = \mathbf{0}$.

$$\begin{pmatrix} 0 & 4/5 & 0 \\ -4/5 & 0 & -3/5 \\ 0 & 3/5 & 0 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & 3/4 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\implies x + \tfrac{3}{4}z = y = 0 \implies \mathbf{x} = t(-3/4, 0, 1).$$

Alternatively, the kernel of L is the set of vectors $\mathbf{v} \in \mathbb{R}^3$ such that $L(\mathbf{v}) = \mathbf{v}_0 \times \mathbf{v} = \mathbf{0}$.
It follows that this is the line spanned by $\mathbf{v}_0 = (3/5, 0, -4/5)$.

Characteristic polynomial of the matrix B:

$$\det(B - \lambda I) = \begin{vmatrix} -\lambda & 4/5 & 0 \\ -4/5 & -\lambda & -3/5 \\ 0 & 3/5 & -\lambda \end{vmatrix}$$

$$= -\lambda^3 - (3/5)^2\lambda - (4/5)^2\lambda = -\lambda^3 - \lambda = -\lambda(\lambda^2 + 1).$$

The eigenvalues are 0, i, and $-i$.

The matrix of the operator L^{2013} is B^{2013}.

Since the matrix B has eigenvalues 0, i, and $-i$, it is diagonalizable in $\mathbb{C}^3$. Namely, $B = UDU^{-1}$, where U is an invertible matrix with complex entries and

$$D = \begin{pmatrix} 0 & 0 & 0 \\ 0 & i & 0 \\ 0 & 0 & -i \end{pmatrix}.$$

Then $B^{2013} = UD^{2013}U^{-1}$. We have that $D^{2013} =$
$= \mathrm{diag}\big(0, i^{2013}, (-i)^{2013}\big) = \mathrm{diag}(0, i, -i) = D$.
Hence

$$B^{2013} = UDU^{-1} = B = \begin{pmatrix} 0 & 4/5 & 0 \\ -4/5 & 0 & -3/5 \\ 0 & 3/5 & 0 \end{pmatrix}.$$