

## Sample problems for Test 1: Solutions

Any problem may be altered or replaced by a different one!

**Problem 1 (15 pts.)** Find a quadratic polynomial  $p(x)$  such that  $p(1) = 1$ ,  $p(2) = 3$ , and  $p(3) = 7$ .

Let  $p(x) = a + bx + cx^2$ . Then  $p(1) = a + b + c$ ,  $p(2) = a + 2b + 4c$ , and  $p(3) = a + 3b + 9c$ . The coefficients  $a$ ,  $b$ , and  $c$  have to be chosen so that

$$\begin{cases} a + b + c = 1, \\ a + 2b + 4c = 3, \\ a + 3b + 9c = 7. \end{cases}$$

We solve this system of linear equations using elementary operations:

$$\begin{aligned} \begin{cases} a + b + c = 1 \\ a + 2b + 4c = 3 \\ a + 3b + 9c = 7 \end{cases} &\iff \begin{cases} a + b + c = 1 \\ b + 3c = 2 \\ a + 3b + 9c = 7 \end{cases} &\iff \begin{cases} a + b + c = 1 \\ b + 3c = 2 \\ 2b + 8c = 6 \end{cases} &\iff \begin{cases} a + b + c = 1 \\ b + 3c = 2 \\ b + 4c = 3 \end{cases} \\ &\iff \begin{cases} a + b + c = 1 \\ b + 3c = 2 \\ c = 1 \end{cases} &\iff \begin{cases} a + b + c = 1 \\ b = -1 \\ c = 1 \end{cases} &\iff \begin{cases} a = 1 \\ b = -1 \\ c = 1 \end{cases} \end{aligned}$$

Thus the desired polynomial is  $p(x) = x^2 - x + 1$ .

**Problem 2 (25 pts.)** Let  $A = \begin{pmatrix} 1 & -2 & 4 & 1 \\ 2 & 3 & 2 & 0 \\ 2 & 0 & -1 & 1 \\ 2 & 0 & 0 & 1 \end{pmatrix}$ .

(i) Evaluate the determinant of the matrix  $A$ .

First let us subtract 2 times the fourth column of  $A$  from the first column:

$$\begin{vmatrix} 1 & -2 & 4 & 1 \\ 2 & 3 & 2 & 0 \\ 2 & 0 & -1 & 1 \\ 2 & 0 & 0 & 1 \end{vmatrix} = \begin{vmatrix} -1 & -2 & 4 & 1 \\ 2 & 3 & 2 & 0 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & 1 \end{vmatrix}.$$

Now the determinant can be easily expanded by the fourth row:

$$\begin{vmatrix} -1 & -2 & 4 & 1 \\ 2 & 3 & 2 & 0 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & 1 \end{vmatrix} = \begin{vmatrix} -1 & -2 & 4 \\ 2 & 3 & 2 \\ 0 & 0 & -1 \end{vmatrix}.$$

The  $3 \times 3$  determinant is easily expanded by the third row:

$$\begin{vmatrix} -1 & -2 & 4 \\ 2 & 3 & 2 \\ 0 & 0 & -1 \end{vmatrix} = (-1) \begin{vmatrix} -1 & -2 \\ 2 & 3 \end{vmatrix}.$$

Thus

$$\det A = - \begin{vmatrix} -1 & -2 \\ 2 & 3 \end{vmatrix} = -1.$$

Another way to evaluate  $\det A$  is to reduce the matrix  $A$  to the identity matrix using elementary row operations (see below). This requires much more work but we are going to do it anyway, to find the inverse of  $A$ .

(ii) Find the inverse matrix  $A^{-1}$ .

First we merge the matrix  $A$  with the identity matrix into one  $4 \times 8$  matrix

$$(A|I) = \left( \begin{array}{cccc|cccc} 1 & -2 & 4 & 1 & 1 & 0 & 0 & 0 \\ 2 & 3 & 2 & 0 & 0 & 1 & 0 & 0 \\ 2 & 0 & -1 & 1 & 0 & 0 & 1 & 0 \\ 2 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \end{array} \right).$$

Then we apply elementary row operations to this matrix until the left part becomes the identity matrix.

Subtract 2 times the first row from the second row:

$$\left( \begin{array}{cccc|cccc} 1 & -2 & 4 & 1 & 1 & 0 & 0 & 0 \\ 2 & 3 & 2 & 0 & 0 & 1 & 0 & 0 \\ 2 & 0 & -1 & 1 & 0 & 0 & 1 & 0 \\ 2 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \end{array} \right) \rightarrow \left( \begin{array}{cccc|cccc} 1 & -2 & 4 & 1 & 1 & 0 & 0 & 0 \\ 0 & 7 & -6 & -2 & -2 & 1 & 0 & 0 \\ 2 & 0 & -1 & 1 & 0 & 0 & 1 & 0 \\ 2 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \end{array} \right).$$

Subtract 2 times the first row from the third row:

$$\left( \begin{array}{cccc|cccc} 1 & -2 & 4 & 1 & 1 & 0 & 0 & 0 \\ 0 & 7 & -6 & -2 & -2 & 1 & 0 & 0 \\ 2 & 0 & -1 & 1 & 0 & 0 & 1 & 0 \\ 2 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \end{array} \right) \rightarrow \left( \begin{array}{cccc|cccc} 1 & -2 & 4 & 1 & 1 & 0 & 0 & 0 \\ 0 & 7 & -6 & -2 & -2 & 1 & 0 & 0 \\ 0 & 4 & -9 & -1 & -2 & 0 & 1 & 0 \\ 2 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \end{array} \right).$$

Subtract 2 times the first row from the fourth row:

$$\left( \begin{array}{cccc|cccc} 1 & -2 & 4 & 1 & 1 & 0 & 0 & 0 \\ 0 & 7 & -6 & -2 & -2 & 1 & 0 & 0 \\ 0 & 4 & -9 & -1 & -2 & 0 & 1 & 0 \\ 2 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \end{array} \right) \rightarrow \left( \begin{array}{cccc|cccc} 1 & -2 & 4 & 1 & 1 & 0 & 0 & 0 \\ 0 & 7 & -6 & -2 & -2 & 1 & 0 & 0 \\ 0 & 4 & -9 & -1 & -2 & 0 & 1 & 0 \\ 0 & 4 & -8 & -1 & -2 & 0 & 0 & 1 \end{array} \right).$$

Subtract 2 times the fourth row from the second row:

$$\left( \begin{array}{cccc|cccc} 1 & -2 & 4 & 1 & 1 & 0 & 0 & 0 \\ 0 & 7 & -6 & -2 & -2 & 1 & 0 & 0 \\ 0 & 4 & -9 & -1 & -2 & 0 & 1 & 0 \\ 0 & 4 & -8 & -1 & -2 & 0 & 0 & 1 \end{array} \right) \rightarrow \left( \begin{array}{cccc|cccc} 1 & -2 & 4 & 1 & 1 & 0 & 0 & 0 \\ 0 & -1 & 10 & 0 & 2 & 1 & 0 & -2 \\ 0 & 4 & -9 & -1 & -2 & 0 & 1 & 0 \\ 0 & 4 & -8 & -1 & -2 & 0 & 0 & 1 \end{array} \right).$$

Subtract the fourth row from the third row:

$$\left( \begin{array}{cccc|cccc} 1 & -2 & 4 & 1 & 1 & 0 & 0 & 0 \\ 0 & -1 & 10 & 0 & 2 & 1 & 0 & -2 \\ 0 & 4 & -9 & -1 & -2 & 0 & 1 & 0 \\ 0 & 4 & -8 & -1 & -2 & 0 & 0 & 1 \end{array} \right) \rightarrow \left( \begin{array}{cccc|cccc} 1 & -2 & 4 & 1 & 1 & 0 & 0 & 0 \\ 0 & -1 & 10 & 0 & 2 & 1 & 0 & -2 \\ 0 & 0 & -1 & 0 & 0 & 0 & 1 & -1 \\ 0 & 4 & -8 & -1 & -2 & 0 & 0 & 1 \end{array} \right).$$

Add 4 times the second row to the fourth row:

$$\left( \begin{array}{cccc|cccc} 1 & -2 & 4 & 1 & 1 & 0 & 0 & 0 \\ 0 & -1 & 10 & 0 & 2 & 1 & 0 & -2 \\ 0 & 0 & -1 & 0 & 0 & 0 & 1 & -1 \\ 0 & 4 & -8 & -1 & -2 & 0 & 0 & 1 \end{array} \right) \rightarrow \left( \begin{array}{cccc|cccc} 1 & -2 & 4 & 1 & 1 & 0 & 0 & 0 \\ 0 & -1 & 10 & 0 & 2 & 1 & 0 & -2 \\ 0 & 0 & -1 & 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 32 & -1 & 6 & 4 & 0 & -7 \end{array} \right).$$

Add 32 times the third row to the fourth row:

$$\left( \begin{array}{cccc|cccc} 1 & -2 & 4 & 1 & 1 & 0 & 0 & 0 \\ 0 & -1 & 10 & 0 & 2 & 1 & 0 & -2 \\ 0 & 0 & -1 & 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 32 & -1 & 6 & 4 & 0 & -7 \end{array} \right) \rightarrow \left( \begin{array}{cccc|cccc} 1 & -2 & 4 & 1 & 1 & 0 & 0 & 0 \\ 0 & -1 & 10 & 0 & 2 & 1 & 0 & -2 \\ 0 & 0 & -1 & 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & -1 & 6 & 4 & 32 & -39 \end{array} \right).$$

Add 10 times the third row to the second row:

$$\left( \begin{array}{cccc|cccc} 1 & -2 & 4 & 1 & 1 & 0 & 0 & 0 \\ 0 & -1 & 10 & 0 & 2 & 1 & 0 & -2 \\ 0 & 0 & -1 & 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & -1 & 6 & 4 & 32 & -39 \end{array} \right) \rightarrow \left( \begin{array}{cccc|cccc} 1 & -2 & 4 & 1 & 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 2 & 1 & 10 & -12 \\ 0 & 0 & -1 & 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & -1 & 6 & 4 & 32 & -39 \end{array} \right).$$

Add the fourth row to the first row:

$$\left( \begin{array}{cccc|cccc} 1 & -2 & 4 & 1 & 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 2 & 1 & 10 & -12 \\ 0 & 0 & -1 & 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & -1 & 6 & 4 & 32 & -39 \end{array} \right) \rightarrow \left( \begin{array}{cccc|cccc} 1 & -2 & 4 & 0 & 7 & 4 & 32 & -39 \\ 0 & -1 & 0 & 0 & 2 & 1 & 10 & -12 \\ 0 & 0 & -1 & 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & -1 & 6 & 4 & 32 & -39 \end{array} \right).$$

Add 4 times the third row to the first row:

$$\left( \begin{array}{cccc|cccc} 1 & -2 & 4 & 0 & 7 & 4 & 32 & -39 \\ 0 & -1 & 0 & 0 & 2 & 1 & 10 & -12 \\ 0 & 0 & -1 & 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & -1 & 6 & 4 & 32 & -39 \end{array} \right) \rightarrow \left( \begin{array}{cccc|cccc} 1 & -2 & 0 & 0 & 7 & 4 & 36 & -43 \\ 0 & -1 & 0 & 0 & 2 & 1 & 10 & -12 \\ 0 & 0 & -1 & 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & -1 & 6 & 4 & 32 & -39 \end{array} \right).$$

Subtract 2 times the second row from the first row:

$$\left( \begin{array}{cccc|cccc} 1 & -2 & 0 & 0 & 7 & 4 & 36 & -43 \\ 0 & -1 & 0 & 0 & 2 & 1 & 10 & -12 \\ 0 & 0 & -1 & 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & -1 & 6 & 4 & 32 & -39 \end{array} \right) \rightarrow \left( \begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 3 & 2 & 16 & -19 \\ 0 & -1 & 0 & 0 & 2 & 1 & 10 & -12 \\ 0 & 0 & -1 & 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & -1 & 6 & 4 & 32 & -39 \end{array} \right).$$

Multiply the second, the third, and the fourth rows by  $-1$ :

$$\left( \begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 3 & 2 & 16 & -19 \\ 0 & -1 & 0 & 0 & 2 & 1 & 10 & -12 \\ 0 & 0 & -1 & 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & -1 & 6 & 4 & 32 & -39 \end{array} \right) \rightarrow \left( \begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 3 & 2 & 16 & -19 \\ 0 & 1 & 0 & 0 & -2 & -1 & -10 & 12 \\ 0 & 0 & 1 & 0 & 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & 1 & -6 & -4 & -32 & 39 \end{array} \right).$$

Finally the left part of our  $4 \times 8$  matrix is transformed into the identity matrix. Therefore the current right part is the inverse matrix of  $A$ . Thus

$$A^{-1} = \begin{pmatrix} 1 & -2 & 4 & 1 \\ 2 & 3 & 2 & 0 \\ 2 & 0 & -1 & 1 \\ 2 & 0 & 0 & 1 \end{pmatrix}^{-1} = \begin{pmatrix} 3 & 2 & 16 & -19 \\ -2 & -1 & -10 & 12 \\ 0 & 0 & -1 & 1 \\ -6 & -4 & -32 & 39 \end{pmatrix}.$$

As a byproduct, we can evaluate the determinant of  $A$ . We have transformed  $A$  into the identity matrix using elementary row operations. These included no row exchanges and three row multiplications, each time by  $-1$ . It follows that  $\det I = (-1)^3 \det A$ . Hence  $\det A = -\det I = -1$ .

**Problem 3 (20 pts.)** Determine which of the following subsets of  $\mathbb{R}^3$  are subspaces. Briefly explain.

- (i) The set  $S_1$  of vectors  $(x, y, z) \in \mathbb{R}^3$  such that  $xyz = 0$ .
- (ii) The set  $S_2$  of vectors  $(x, y, z) \in \mathbb{R}^3$  such that  $x + y + z = 0$ .
- (iii) The set  $S_3$  of vectors  $(x, y, z) \in \mathbb{R}^3$  such that  $y^2 + z^2 = 0$ .
- (iv) The set  $S_4$  of vectors  $(x, y, z) \in \mathbb{R}^3$  such that  $y^2 - z^2 = 0$ .

A subset of  $\mathbb{R}^3$  is a subspace if it is closed under addition and scalar multiplication. Besides, a subspace must not be empty.

It is easy to see that each of the sets  $S_1$ ,  $S_2$ ,  $S_3$ , and  $S_4$  contains the zero vector  $(0, 0, 0)$  and all these sets are closed under scalar multiplication.

The set  $S_1$  is the union of three planes  $x = 0$ ,  $y = 0$ , and  $z = 0$ . It is not closed under addition as the following example shows:  $(1, 1, 0) + (0, 0, 1) = (1, 1, 1)$ .

$S_2$  is a plane passing through the origin. Obviously, it is closed under addition.

The condition  $y^2 + z^2 = 0$  is equivalent to  $y = z = 0$ . Hence  $S_3$  is a line passing through the origin. It is closed under addition.

Since  $y^2 - z^2 = (y - z)(y + z)$ , the set  $S_4$  is the union of two planes  $y - z = 0$  and  $y + z = 0$ . The following example shows that  $S_4$  is not closed under addition:  $(0, 1, 1) + (0, 1, -1) = (0, 2, 0)$ .

Thus  $S_2$  and  $S_3$  are subspaces of  $\mathbb{R}^3$  while  $S_1$  and  $S_4$  are not.

**Problem 4 (30 pts.)** Let  $B = \begin{pmatrix} 0 & -1 & 4 & 1 \\ 1 & 1 & 2 & -1 \\ -3 & 0 & -1 & 0 \\ 2 & -1 & 0 & 1 \end{pmatrix}$ .

- (i) Find the rank and the nullity of the matrix  $B$ .

The rank (dimension of the row space) and the nullity (dimension of the nullspace) of a matrix are preserved under elementary row operations. We apply such operations to convert the matrix  $B$  into row echelon form.

First interchange the first row with the second row:

$$\begin{pmatrix} 0 & -1 & 4 & 1 \\ 1 & 1 & 2 & -1 \\ -3 & 0 & -1 & 0 \\ 2 & -1 & 0 & 1 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 1 & 2 & -1 \\ 0 & -1 & 4 & 1 \\ -3 & 0 & -1 & 0 \\ 2 & -1 & 0 & 1 \end{pmatrix}.$$

Add 3 times the first row to the third row, then subtract 2 times the first row from the fourth row:

$$\begin{pmatrix} 1 & 1 & 2 & -1 \\ 0 & -1 & 4 & 1 \\ -3 & 0 & -1 & 0 \\ 2 & -1 & 0 & 1 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 1 & 2 & -1 \\ 0 & -1 & 4 & 1 \\ 0 & 3 & 5 & -3 \\ 2 & -1 & 0 & 1 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 1 & 2 & -1 \\ 0 & -1 & 4 & 1 \\ 0 & 3 & 5 & -3 \\ 0 & -3 & -4 & 3 \end{pmatrix}.$$

Multiply the second row by  $-1$ :

$$\begin{pmatrix} 1 & 1 & 2 & -1 \\ 0 & -1 & 4 & 1 \\ 0 & 3 & 5 & -3 \\ 0 & -3 & -4 & 3 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 1 & 2 & -1 \\ 0 & 1 & -4 & -1 \\ 0 & 3 & 5 & -3 \\ 0 & -3 & -4 & 3 \end{pmatrix}.$$

Add the fourth row to the third row:

$$\begin{pmatrix} 1 & 1 & 2 & -1 \\ 0 & 1 & -4 & -1 \\ 0 & 3 & 5 & -3 \\ 0 & -3 & -4 & 3 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 1 & 2 & -1 \\ 0 & 1 & -4 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & -3 & -4 & 3 \end{pmatrix}.$$

Add 3 times the second row to the fourth row:

$$\begin{pmatrix} 1 & 1 & 2 & -1 \\ 0 & 1 & -4 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & -3 & -4 & 3 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 1 & 2 & -1 \\ 0 & 1 & -4 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & -16 & 0 \end{pmatrix}.$$

Add 16 times the third row to the fourth row:

$$\begin{pmatrix} 1 & 1 & 2 & -1 \\ 0 & 1 & -4 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & -16 & 0 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 1 & 2 & -1 \\ 0 & 1 & -4 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Now that the matrix is in row echelon form, its rank equals the number of nonzero rows, which is 3. Since

$$(\text{rank of } B) + (\text{nullity of } B) = (\text{the number of columns of } B) = 4,$$

it follows that the nullity of  $B$  equals 1.

(ii) Find a basis for the row space of  $B$ , then extend this basis to a basis for  $\mathbb{R}^4$ .

The row space of a matrix is invariant under elementary row operations. Therefore the row space of the matrix  $B$  is the same as the row space of its row echelon form

$$\begin{pmatrix} 1 & 1 & 2 & -1 \\ 0 & 1 & -4 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

The nonzero rows of the latter are linearly independent so that they form a basis for its row space. Hence the vectors  $\mathbf{v}_1 = (1, 1, 2, -1)$ ,  $\mathbf{v}_2 = (0, 1, -4, -1)$ , and  $\mathbf{v}_3 = (0, 0, 1, 0)$  form a basis for the row space of  $B$ .

To extend the basis  $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$  to a basis for  $\mathbb{R}^4$ , we need a vector  $\mathbf{v}_4 \in \mathbb{R}^4$  that is not a linear combination of  $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ . It is known that at least one of the vectors  $\mathbf{e}_1 = (1, 0, 0, 0)$ ,  $\mathbf{e}_2 = (0, 1, 0, 0)$ ,  $\mathbf{e}_3 = (0, 0, 1, 0)$ , and  $\mathbf{e}_4 = (0, 0, 0, 1)$  can be chosen as  $\mathbf{v}_4$ . In particular, the vectors  $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{e}_4$  form a basis for  $\mathbb{R}^4$ . This follows from the fact that the  $4 \times 4$  matrix whose rows are these vectors is not singular:

$$\begin{vmatrix} 1 & 1 & 2 & -1 \\ 0 & 1 & -4 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{vmatrix} = 1 \neq 0.$$

(iii) Find a basis for the nullspace of  $B$ .

The nullspace of  $B$  is the solution set of the system of linear homogeneous equations with  $B$  as the coefficient matrix. To solve the system, we convert the matrix  $B$  to reduced row echelon form. The row echelon form of  $B$  has been obtained earlier:

$$\begin{pmatrix} 1 & 1 & 2 & -1 \\ 0 & 1 & -4 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Add 4 times the third row to the second row, then subtract 2 times the third row from the first row:

$$\begin{pmatrix} 1 & 1 & 2 & -1 \\ 0 & 1 & -4 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 1 & 2 & -1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 1 & 0 & -1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Subtract the second row from the first row:

$$\begin{pmatrix} 1 & 1 & 0 & -1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

We have obtained the reduced row echelon form of the matrix  $B$ . Its nullspace is the same as the nullspace of  $B$ . Hence a vector  $(x_1, x_2, x_3, x_4) \in \mathbb{R}^4$  belongs to the nullspace of  $B$  if and only if

$$\begin{cases} x_1 = 0, \\ x_2 - x_4 = 0, \\ x_3 = 0 \end{cases} \iff \begin{cases} x_1 = 0, \\ x_2 = x_4, \\ x_3 = 0. \end{cases}$$

The general solution of this system is  $(x_1, x_2, x_3, x_4) = (0, t, 0, t) = t(0, 1, 0, 1)$ ,  $t \in \mathbb{R}$ . Thus the nullspace of the matrix  $B$  is spanned by the vector  $(0, 1, 0, 1)$ . This vector forms a basis for the nullspace.

**Bonus Problem 5 (15 pts.)** Show that the functions  $f_1(x) = x$ ,  $f_2(x) = xe^x$ , and  $f_3(x) = e^{-x}$  are linearly independent in the vector space  $C^\infty(\mathbb{R})$ .

Suppose that  $af_1(x) + bf_2(x) + cf_3(x) = 0$  for all  $x \in \mathbb{R}$ , where  $a, b, c$  are constants. We have to show that  $a = b = c = 0$ .

Differentiating the identity  $af_1(x) + bf_2(x) + cf_3(x) = 0$  four times, we obtain four more identities:

$$\begin{aligned} ax + bxe^x + ce^{-x} &= 0, \\ a + be^x + bxe^x - ce^{-x} &= 0, \\ 2be^x + bxe^x + ce^{-x} &= 0, \\ 3be^x + bxe^x - ce^{-x} &= 0, \\ 4be^x + bxe^x + ce^{-x} &= 0. \end{aligned}$$

Subtracting the third identity from the fifth one, we obtain  $2be^x = 0$ , which implies that  $b = 0$ . Substituting  $b = 0$  in the third identity, we obtain  $ce^{-x} = 0$ , which implies that  $c = 0$ . Substituting  $b = 0$  and  $c = 0$  in the second identity, we obtain  $a = 0$ .

*Alternative solution:* Suppose that  $ax + bxe^x + ce^{-x} = 0$  for all  $x \in \mathbb{R}$ , where  $a, b, c$  are constants. We have to show that  $a = b = c = 0$ .

For any  $x \neq 0$  divide both sides of the identity by  $xe^x$ :

$$ae^{-x} + b + cx^{-1}e^{-2x} = 0.$$

Note that  $e^{-x} \rightarrow 0$  and  $x^{-1}e^{-2x} \rightarrow 0$  as  $x \rightarrow +\infty$ . Hence the left-hand side approaches  $b$  as  $x \rightarrow +\infty$ . It follows that  $b = 0$ . Now  $ax + ce^{-x} = 0$  for all  $x \in \mathbb{R}$ . For any  $x \neq 0$  divide both sides of the latter identity by  $x$ :

$$a + cx^{-1}e^{-x} = 0.$$

Since  $x^{-1}e^{-x} \rightarrow 0$  as  $x \rightarrow +\infty$ , the left-hand side approaches  $a$  as  $x \rightarrow +\infty$ . It follows that  $a = 0$ . Then  $ce^{-x} = 0$ , which implies that  $c = 0$ .

**Bonus Problem 6 (15 pts.)** Let  $V$  be a finite-dimensional vector space and  $V_0$  be a proper subspace of  $V$  (where proper means that  $V_0 \neq V$ ). Prove that  $\dim V_0 < \dim V$ .

Any linearly independent set in a vector space can be extended to a basis. Since the vector space  $V$  is finite dimensional, it does not admit infinitely many linearly independent vectors. Clearly, the same is true for the subspace  $V_0$ . It follows that  $V_0$  is also finite-dimensional.

Let  $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_k$  be a basis for  $V_0$ . The vectors  $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_k$  are linearly independent in  $V$  since they are linearly independent in  $V_0$ . Therefore we can extend this collection of vectors to a basis for  $V$  by adding some vectors  $\mathbf{w}_1, \dots, \mathbf{w}_m$ . As  $V_0 \neq V$ , we do need to add some vectors, i.e.,  $m \geq 1$ . Thus  $\dim V_0 = k$  and  $\dim V = k + m > k$ .