MATH 409

Advanced Calculus I

Lecture 3:
Metric spaces.
Completeness axiom.
Existence of square roots.

Absolute value

Definition. The **absolute value** (or **modulus**) of a real number a, denoted $|a|$, is defined as follows:

$$|a| = \begin{cases} a & \text{if } \ a \geq 0, \\ -a & \text{if } \ a < 0. \end{cases}$$

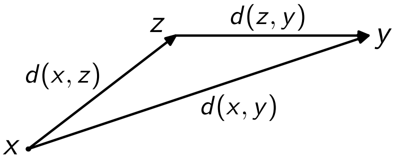
Properties of the absolute value:

- $|a| \geq 0$;
- $|a| = 0$ if and only if $a = 0$;
- $|-a| = |a|$;
- If $M > 0$, then $|a| < M \iff -M < a < M$;
- $|ab| = |a| \cdot |b|$;
- $|a + b| \leq |a| + |b|$.

Metric space

Definition. Given a nonempty set X, a **metric** (or **distance function**) on X is a function $d : X \times X \to \mathbb{R}$ that satisfies the following conditions:

- **(positivity)** $d(x, y) \geq 0$ for all $x, y \in X$; moreover, $d(x, y) = 0$ if and only if $x = y$;
- **(symmetry)** $d(x, y) = d(y, x)$ for all $x, y \in X$;
- **(triangle inequality)** $d(x, y) \leq d(x, z) + d(z, y)$ for all $x, y, z \in X$.



A set endowed with a metric is called a **metric space**.

Theorem The function $d(x, y) = |y - x|$ is a metric on the real line $\mathbb{R}$.

Proof: We have $|y - x| \geq 0$ for all $x, y \in \mathbb{R}$. Moreover, $|y - x| = 0$ only if $y - x = 0$, which is equivalent to $x = y$. This proves positivity.

Symmetry follows since $x - y = -(y - x)$ and $|-a| = |a|$ for all $a \in \mathbb{R}$.

Finally, $d(x, y) = |y - x| = |(y - z) + (z - x)|$
$\leq |y - z| + |z - x| = d(z, y) + d(x, z)$.

Other examples of metric spaces

- *Euclidean space*

$X = \mathbb{R}^n$, $d(\mathbf{x}, \mathbf{y}) = \sqrt{(y_1 - x_1)^2 + (y_2 - x_2)^2 + \cdots + (y_n - x_n)^2}$.

- *Normed vector space*

X: vector space with a norm $\|\cdot\|$, $d(\mathbf{x}, \mathbf{y}) = \|\mathbf{y} - \mathbf{x}\|$.

- *Discrete metric space*

X: any nonempty set, $d(x, y) = 1$ if $x \neq y$ and $d(x, y) = 0$ if $x = y$.

- *Space of sequences*

X: set of all infinite words $x = x_1 x_2 \dots$ over a finite alphabet; $d(x, y) = 2^{-n}$ if $x_i = y_i$ for $1 \leq i \leq n$ while $x_{n+1} \neq y_{n+1}$, $d(x, y) = 0$ if $x_i = y_i$ for all $i \geq 1$.

Supremum and infimum

Definition. Let $E \subset \mathbb{R}$ be a nonempty set and M be a real number. We say that M is an **upper bound** of the set E if $a \leq M$ for all $a \in E$. Similarly, M is a **lower bound** of the set E if $a \geq M$ for all $a \in E$.

We say that the set E is **bounded above** if it admits an upper bound and **bounded below** if it admits a lower bound. The set E is called **bounded** if it is bounded above and below.

A real number M is called the **supremum** (or the **least upper bound**) of the set E and denoted $\sup E$ if (i) M is an upper bound of E and (ii) $M \leq M_+$ for any upper bound M_+ of E.

Similarly, M is called the **infimum** (or the **greatest lower bound**) of the set E and denoted $\inf E$ if (i) M is a lower bound of E and (ii) $M \geq M_-$ for any lower bound M_- of E.

Axioms of real numbers

Definition. The set $\mathbb{R}$ of real numbers is a set satisfying the following postulates:

Postulate 1. $\mathbb{R}$ is a field.

Postulate 2. There is a strict linear order $<$ on $\mathbb{R}$ that makes it into an ordered field.

Postulate 3 (Completeness Axiom).
If a nonempty subset $E \subset \mathbb{R}$ is bounded above, then E has a supremum.

Theorem 1 Suppose X and Y are nonempty subsets of $\mathbb{R}$ such that $a \leq b$ for all $a \in X$ and $b \in Y$. Then there exists $c \in \mathbb{R}$ such that $a \leq c$ for all $a \in X$ and $c \leq b$ for all $b \in Y$.

Proof: The set X is bounded above as any element of Y is an upper bound of X. By Completeness Axiom, $\sup X$ exists. We have $a \leq \sup X$ for all $a \in X$ since $\sup X$ is an upper bound of X. Besides, $\sup X \leq b$ for any $b \in Y$ since b is an upper bound of X while $\sup X$ is the least upper bound.

Theorem 2 If a nonempty subset $E \subset \mathbb{R}$ is bounded below, then E has an infimum.

Proof: Let X denote the set of all lower bounds of E. Then $a \leq b$ for all $a \in X$ and $b \in E$. Since E is bounded below, the set X is not empty. By Theorem 1, there exists $c \in \mathbb{R}$ such that $a \leq c$ for all $a \in X$ and $c \leq b$ for all $b \in E$. That is, c is a lower bound of E and an upper bound of X. It follows that $c = \inf E$.

Natural, integer, and rational numbers

Postulate 1 guarantees that $\mathbb{R}$ contains numbers 0 and 1. Then we can define natural numbers $2 = 1 + 1$, $3 = 2 + 1$, $4 = 3 + 1$, and so on... It was proved in the previous lecture that $0 < 1$. Repeatedly adding 1 to both sides of this inequality, we obtain $0 < 1 < 2 < 3 < \dots$ In particular, all these numbers are distinct.

However the entire set of natural numbers can only be defined in an implicit way.

Definition. A set $E \subset \mathbb{R}$ is called **inductive** if $1 \in E$ and, for any real number x, $x \in E$ implies $x + 1 \in E$. The set $\mathbb{N}$ of **natural numbers** is the smallest inductive subset of $\mathbb{R}$ (namely, it is the intersection of all inductive subsets of $\mathbb{R}$).

The set of **integers** is defined as $\mathbb{Z} = -\mathbb{N} \cup \{0\} \cup \mathbb{N}$.
The set of **rationals** is defined as $\mathbb{Q} = \{m/n \mid m \in \mathbb{Z}, n \in \mathbb{N}\}$.

Archimedean Principle

Theorem (Archimedean Principle) For any real number $\varepsilon > 0$ there exists a natural number n such that $n\varepsilon > 1$.

Remark. Archimedean Principle means that $\mathbb{R}$ contains no **infinitesimal** (i.e., infinitely small) numbers other than 0.

Proof: In the case $\varepsilon > 1$, we can take $n = 1$. Now assume $\varepsilon \leq 1$. Let E be the set of all natural numbers n such that $n\varepsilon \leq 1$. Observe that E is nonempty ($1 \in E$) and bounded above ($1/\varepsilon$ is an upper bound). By Completeness Axiom, $m = \sup E$ exists. By definition of $\sup E$, there exists $n \in E$ such that $n > m - 1/2$ (as otherwise $m - 1/2$ would be an upper bound for E). Then $n + 1$ is a natural number and $n + 1 > m + 1/2 > m$. It follows that $n + 1$ is not in E. Consequently, $(n + 1)\varepsilon > 1$. ∎

Corollary For any $a, b > 0$ there exists a natural number n such that $na > b$.

Density of rational numbers

Theorem For any real numbers a and b, $a < b$, there exists a rational number ξ such that $a < \xi < b$.

Proof: By Archimedean Principle, there exists a natural number n such that $n(b - a) > 1$. Let E be the set of all integers m such that $m/n < b$. Observe that E is bounded above (nb is an upper bound). Let us show that the set E is not empty. In the case $b \geq 0$ it is obvious as $-1 \in E$. In the case $b < 0$, we have $-b > 0$. By Archimedean Principle, there exists a natural number m such that $m(-nb)^{-1} > 1$. Then $-m/n < b$ so that $-m \in E$.

By Completeness Axiom, $k = \sup E$ exists. By definition of $\sup E$, there exists $m \in E$ such that $m > k - 1/2$. Then $m + 1$ is an integer and $m + 1 > k + 1/2 > k$, which implies that $m + 1$ is not in E. Therefore $m/n < b \leq (m + 1)/n$. Consequently, $m/n \geq b - 1/n > b - (b - a) = a$. Thus $a < m/n < b$.

Existence of square roots

Theorem For any $a > 0$ there exists a unique number $r > 0$ (denoted $\sqrt{a}$) such that $r^2 = a$.

We begin the proof with the following simple lemmas.

Lemma 1 Suppose r and t are positive numbers. Then $r^2 < t^2$ if and only if $r < t$.

Lemma 2 Suppose r and t are positive numbers. Then $r^2 = t^2$ if and only if $r = t$.

Proof of Lemmas 1 and 2: By linearity of the order on $\mathbb{R}$, we have either $r < t$ or $r > t$ or $r = t$. Since $r, t > 0$, we obtain that $r < t \implies r^2 < t^2$ and $r > t \implies r^2 > t^2$. Besides, $r = t \implies r^2 = t^2$. We conclude that $r^2 < t^2$ if and only if $r < t$. Also, $r^2 = t^2$ if and only if $r = t$. ∎

Lemma 2 immediately implies uniqueness of $\sqrt{a}$.

To prove existence of the square root $\sqrt{a}$, let us consider a set $E = \{x > 0 \mid x^2 < a\}$. We shall show that $r = \sup E$ is the desired number. First we need to verify that $\sup E$ exists. By Completeness Axiom, it is enough to check that the set E is nonempty and bounded above. Moreover, Lemma 1 implies that any $b > 0$ satisfying $a \leq b^2$ is an upper bound of E.

Consider three cases: $a > 1$, $a < 1$, and $a = 1$.

If $a > 1$ then $1 \in E$. Also, $a < a^2$ so that a is an upper bound of E. If $a < 1$ then $a^2 < a$ so that $a \in E$. Also, 1 is an upper bound for E. If $a = 1$, then $1/2 \in E$ and 1 is an upper bound of E.

Thus $r = \sup E$ exists. Clearly, $r > 0$. We claim that $r^2 = a$. Assume the contrary. Then $r^2 < a$ or $r^2 > a$.
In the 1st case, there is no $t > 0$ such that $r^2 < t^2 < a$.
In the 2nd case, there is no $t > 0$ such that $a < t^2 < r^2$.
Now we get a contradiction once the following lemma is proved:

Lemma 3 Suppose a and r are positive real numbers and $a \neq r^2$. Then there exists $t > 0$ such that t^2 lies between a and r^2, i.e., $a < t^2 < r^2$ or $r^2 < t^2 < a$.

Proof: First we consider a special case when $0 < a < 1$ and $r = 1$. Let us show that $t = (1 + a)/2$ is a desired number in this case. Indeed, $0 < a < 1$ implies that $1 < 1 + a < 2$, then $0 < t < 1$ and $t^2 < t < 1$. Further, $4(t^2 - a) =$
$= (2t)^2 - 4a = (1+a)^2 - 4a = (1+2a+a^2) - 4a = 1 - 2a + a^2$
$= (1 - a)^2 > 0$ since $1 - a > 0$. Hence $a < t^2 < 1 = r^2$.

Next we consider a more general case $a < r^2$. In this case, $0 < ar^{-2} < 1$, where $r^{-2} = (r^2)^{-1}$, which is also $(r^{-1})^2$. By the above there exists $t > 0$ such that $ar^{-2} < t^2 < 1$. Then tr is a positive number and $a < t^2 r^2 = (tr)^2 < r^2$.

It remains to consider the case $r^2 < a$. In this case, $0 < a^{-1} < r^{-2} = (r^{-1})^2$. By the above there exists $t > 0$ such that $a^{-1} < t^2 < r^{-2}$. Then t^{-1} is a positive number and $r^2 < t^{-2} = (t^{-1})^2 < a$. ■