MATH 423

Linear Algebra II

Lecture 2:
Vector spaces: examples and basic properties.

Vector space

A *vector space* is a set V equipped with two operations, **addition**

$$V \times V \ni (\mathbf{x}, \mathbf{y}) \mapsto \mathbf{x} + \mathbf{y} \in V$$

and **scalar multiplication**

$$\mathbb{R} \times V \ni (r, \mathbf{x}) \mapsto r\mathbf{x} \in V,$$

that have the following properties:

Properties of addition and scalar multiplication

VS1. $\mathbf{x}+\mathbf{y}=\mathbf{y}+\mathbf{x}$ for all $\mathbf{x},\mathbf{y}\in V$.

VS2. $(\mathbf{x}+\mathbf{y})+\mathbf{z}=\mathbf{x}+(\mathbf{y}+\mathbf{z})$ for all $\mathbf{x},\mathbf{y},\mathbf{z}\in V$.

VS3. There exists an element of V, called the *zero vector* and denoted $\mathbf{0}$, such that $\mathbf{x}+\mathbf{0}=\mathbf{0}+\mathbf{x}=\mathbf{x}$ for all $\mathbf{x}\in V$.

VS4. For any $\mathbf{x}\in V$ there exists an element of V, denoted $-\mathbf{x}$, such that $\mathbf{x}+(-\mathbf{x})=(-\mathbf{x})+\mathbf{x}=\mathbf{0}$.

VS5. $1\mathbf{x}=\mathbf{x}$ for all $\mathbf{x}\in V$.

VS6. $(rs)\mathbf{x}=r(s\mathbf{x})$ for all $r,s\in\mathbb{R}$ and $\mathbf{x}\in V$.

VS7. $r(\mathbf{x}+\mathbf{y})=r\mathbf{x}+r\mathbf{y}$ for all $r\in\mathbb{R}$ and $\mathbf{x},\mathbf{y}\in V$.

VS8. $(r+s)\mathbf{x}=r\mathbf{x}+s\mathbf{x}$ for all $r,s\in\mathbb{R}$ and $\mathbf{x}\in V$.

- Associativity of addition implies that a multiple sum $\mathbf{u}_1 + \mathbf{u}_2 + \cdots + \mathbf{u}_k$ is well defined for any $\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_k \in V$.

- **Subtraction** in V is defined as follows: $\mathbf{x} - \mathbf{y} = \mathbf{x} + (-\mathbf{y})$.

- Addition and scalar multiplication are called **linear operations**.

Given $\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_k \in V$ and $r_1, r_2, \ldots, r_k \in \mathbb{R}$,

$$\boxed{r_1\mathbf{u}_1 + r_2\mathbf{u}_2 + \cdots + r_k\mathbf{u}_k}$$

is called a **linear combination** of $\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_k$.

Examples of vector spaces

In most examples, addition and scalar multiplication are natural operations so that properties VS1–VS8 are easy to verify.

- $\mathbb{R}^n$: n-dimensional coordinate vectors
- $\mathcal{M}_{m,n}(\mathbb{R})$: $m \times n$ matrices with real entries

- $\mathbb{R}^\infty$: infinite sequences $(x_1, x_2, \dots)$ of real numbers (also denoted $\{x_n\}$)

For any $\mathbf{x} = (x_1, x_2, \dots)$, $\mathbf{y} = (y_1, y_2, \dots) \in \mathbb{R}^\infty$ and $r \in \mathbb{R}$
let $\mathbf{x} + \mathbf{y} = (x_1 + y_1, x_2 + y_2, \dots)$, $r\mathbf{x} = (rx_1, rx_2, \dots)$.
Then $\mathbf{0} = (0, 0, \dots)$ and $-\mathbf{x} = (-x_1, -x_2, \dots)$.

- $\{\mathbf{0}\}$: the zero (or trivial) vector space

$\mathbf{0} + \mathbf{0} = \mathbf{0}$, $r\mathbf{0} = \mathbf{0}$, $-\mathbf{0} = \mathbf{0}$.

Linear operations on matrices

An $m \times n$ *matrix* is a rectangular array of numbers (called *entries*) that has m rows and n columns. Any element of $\mathbb{R}^n$ may be regarded as an $n \times 1$ matrix (column vector) or as a $1 \times n$ matrix (row vector).

If A is a matrix, then A_{ij} denotes its entry in row i and column j. Alternative notation: $A = (a_{ij})$, where a_{ij} is the entry in row i and column j. Let $A, B \in \mathcal{M}_{m,n}(\mathbb{R})$ and $r \in \mathbb{R}$.

Matrix sum: $(A + B)_{ij} = A_{ij} + B_{ij}$

Scalar multiple: $(rA)_{ij} = rA_{ij}$

Zero matrix O: all entries are zeros

Negative of a matrix: $(-A)_{ij} = -A_{ij}$

As far as the linear operations are concerned, the $m \times n$ matrices have the same properties as vectors in $\mathbb{R}^{mn}$.

Functional vector spaces

- $\mathcal{F}(S)$: the set of all functions $f : S \to \mathbb{R}$,

where S is a nonempty set.

Given functions $f, g \in \mathcal{F}(S)$ and a scalar $r \in \mathbb{R}$, let $(f+g)(x) = f(x) + g(x)$ and $(rf)(x) = rf(x)$ for all $x \in S$.
Zero vector: $o(x) = 0$. Negative: $(-f)(x) = -f(x)$.

- $C(\mathbb{R})$: all continuous functions $f : \mathbb{R} \to \mathbb{R}$

Linear operations are inherited from $\mathcal{F}(\mathbb{R})$. We only need to check that $f, g \in C(\mathbb{R}) \implies f+g, rf \in C(\mathbb{R})$, the zero function is continuous, and $f \in C(\mathbb{R}) \implies -f \in C(\mathbb{R})$.

- $C^1(\mathbb{R})$: all continuously differentiable functions

$f : \mathbb{R} \to \mathbb{R}$

- $C^\infty(\mathbb{R})$: all smooth functions $f : \mathbb{R} \to \mathbb{R}$
- $\mathcal{P}$: all polynomials $p(x) = a_0 + a_1 x + \cdots + a_n x^n$

Counterexample: dumb scaling

Consider the set $V = \mathbb{R}^2$ with the standard addition and a nonstandard scalar multiplication:

$$\boxed{r \odot \mathbf{x} = \mathbf{0}} \quad \text{for any } \mathbf{x} \in \mathbb{R}^2 \text{ and } r \in \mathbb{R}.$$

Properties VS1–VS4 still hold because they do not involve scalar multiplication.

VS5. $1 \odot \mathbf{x} = \mathbf{x} \iff \mathbf{0} = \mathbf{x}$

VS6. $(rs) \odot \mathbf{x} = r \odot (s \odot \mathbf{x}) \iff \mathbf{0} = \mathbf{0}$

VS7. $r \odot (\mathbf{x} + \mathbf{y}) = r \odot \mathbf{x} + r \odot \mathbf{y} \iff \mathbf{0} = \mathbf{0} + \mathbf{0}$

VS8. $(r + s) \odot \mathbf{x} = r \odot \mathbf{x} + s \odot \mathbf{x} \iff \mathbf{0} = \mathbf{0} + \mathbf{0}$

VS5 is the only property that fails. Therefore property VS5 does not follow from the others.

Counterexample: lazy scaling

Consider the set $V = \mathbb{R}^2$ with the standard addition and a nonstandard scalar multiplication:

$$\boxed{r \odot \mathbf{x} = \mathbf{x}} \quad \text{for any } \mathbf{x} \in \mathbb{R}^2 \text{ and } r \in \mathbb{R}.$$

Properties VS1–VS4 still hold because they do not involve scalar multiplication.

VS5. $1 \odot \mathbf{x} = \mathbf{x} \iff \mathbf{x} = \mathbf{x}$

VS6. $(rs) \odot \mathbf{x} = r \odot (s \odot \mathbf{x}) \iff \mathbf{x} = \mathbf{x}$

VS7. $r \odot (\mathbf{x} + \mathbf{y}) = r \odot \mathbf{x} + r \odot \mathbf{y} \iff \mathbf{x} + \mathbf{y} = \mathbf{x} + \mathbf{y}$

VS8. $(r + s) \odot \mathbf{x} = r \odot \mathbf{x} + s \odot \mathbf{x} \iff \mathbf{x} = \mathbf{x} + \mathbf{x}$

The only property that fails is VS8.

Weird example

Consider the set $V = \mathbb{R}_+$ of positive numbers with a nonstandard addition and scalar multiplication:

$$\boxed{x \oplus y = xy} \quad \text{for any } x, y \in \mathbb{R}_+.$$

$$\boxed{r \odot x = x^r} \quad \text{for any } x \in \mathbb{R}_+ \text{ and } r \in \mathbb{R}.$$

VS1. $x \oplus y = y \oplus x \iff xy = yx$

VS2. $(x \oplus y) \oplus z = x \oplus (y \oplus z) \iff (xy)z = x(yz)$

VS3. $x \oplus \zeta = \zeta \oplus x = x \iff x\zeta = \zeta x = x$ (holds for $\zeta = 1$)

VS4. $x \oplus \eta = \eta \oplus x = 1 \iff x\eta = \eta x = 1$ (holds for $\eta = x^{-1}$)

VS5. $1 \odot x = x \iff x^1 = x$

VS6. $(rs) \odot x = r \odot (s \odot x) \iff x^{rs} = (x^s)^r$

VS7. $r \odot (x \oplus y) = (r \odot x) \oplus (r \odot y) \iff (xy)^r = x^r y^r$

VS8. $(r + s) \odot x = (r \odot x) \oplus (s \odot x) \iff x^{r+s} = x^r x^s$

Some general observations

- The zero vector is unique.

Suppose $\mathbf{z}_1$ and $\mathbf{z}_2$ are zero vectors. Then $\mathbf{z}_1 + \mathbf{z}_2 = \mathbf{z}_2$ since $\mathbf{z}_1$ is a zero vector and $\mathbf{z}_1 + \mathbf{z}_2 = \mathbf{z}_1$ since $\mathbf{z}_2$ is a zero vector. Hence $\mathbf{z}_1 = \mathbf{z}_2$.

- For any $\mathbf{x} \in V$, the negative $-\mathbf{x}$ is unique.

Suppose $\mathbf{y}$ and $\mathbf{y}'$ are additive inverses of $\mathbf{x}$. Let us compute the sum $\mathbf{y}' + \mathbf{x} + \mathbf{y}$ in two ways:

$$(\mathbf{y}' + \mathbf{x}) + \mathbf{y} = \mathbf{0} + \mathbf{y} = \mathbf{y},$$
$$\mathbf{y}' + (\mathbf{x} + \mathbf{y}) = \mathbf{y}' + \mathbf{0} = \mathbf{y}'.$$

By associativity of the vector addition, $\mathbf{y} = \mathbf{y}'$.

Some general observations

- (cancellation law) $\mathbf{x} + \mathbf{y} = \mathbf{x}' + \mathbf{y}$ implies $\mathbf{x} = \mathbf{x}'$ for any $\mathbf{x}, \mathbf{x}', \mathbf{y} \in V$.

If $\mathbf{x} + \mathbf{y} = \mathbf{x}' + \mathbf{y}$ then $(\mathbf{x} + \mathbf{y}) + (-\mathbf{y}) = (\mathbf{x}' + \mathbf{y}) + (-\mathbf{y})$. By associativity, $(\mathbf{x} + \mathbf{y}) + (-\mathbf{y}) = \mathbf{x} + (\mathbf{y} + (-\mathbf{y})) = \mathbf{x} + \mathbf{0} = \mathbf{x}$ and $(\mathbf{x}' + \mathbf{y}) + (-\mathbf{y}) = \mathbf{x}' + (\mathbf{y} + (-\mathbf{y})) = \mathbf{x}' + \mathbf{0} = \mathbf{x}'$. Hence $\mathbf{x} = \mathbf{x}'$.

- $0\mathbf{x} = \mathbf{0}$ for any $\mathbf{x} \in V$.

Indeed, $0\mathbf{x} + \mathbf{x} = 0\mathbf{x} + 1\mathbf{x} = (0 + 1)\mathbf{x} = 1\mathbf{x} = \mathbf{x} = \mathbf{0} + \mathbf{x}$.
By the cancellation law, $0\mathbf{x} = \mathbf{0}$.

- $(-1)\mathbf{x} = -\mathbf{x}$ for any $\mathbf{x} \in V$.

Indeed, $\mathbf{x} + (-1)\mathbf{x} = (-1)\mathbf{x} + \mathbf{x} = (-1)\mathbf{x} + 1\mathbf{x} = (-1 + 1)\mathbf{x}$
$= 0\mathbf{x} = \mathbf{0}$.