MATH 423 Linear Algebra II Lecture 10: Inverse matrix. Change of coordinates. Let V be a vector space and $\alpha = [\mathbf{v}_1, \dots, \mathbf{v}_n]$ be an ordered basis for V.

Theorem 1 The coordinate mapping $C: V \to \mathbb{F}^n$ given by $C(\mathbf{v}) = [\mathbf{v}]_{\alpha}$ is linear and invertible (i.e., one-to-one and onto).

Let W be another vector space and $\beta = [\mathbf{w}_1, \dots, \mathbf{w}_m]$ be an ordered basis for W.

Theorem 2 The mapping $M : \mathcal{L}(V, W) \to \mathcal{M}_{m,n}(\mathbb{F})$ given by $M(L) = [L]^{\beta}_{\alpha}$ is linear and invertible.

Linear maps and matrix multiplication

Let V, W, and X be vector spaces. Suppose $\alpha = [\mathbf{v}_1, \dots, \mathbf{v}_n]$ is an ordered basis for V, $\beta = [\mathbf{w}_1, \dots, \mathbf{w}_m]$ is an ordered basis for W, and $\gamma = [\mathbf{x}_1, \dots, \mathbf{x}_k]$ is an ordered basis for X.

Theorem 1 For any linear transformation $L: V \to W$ and any vector $\mathbf{v} \in V$, $[L(\mathbf{v})]_{\beta} = [L]_{\alpha}^{\beta} [\mathbf{v}]_{\alpha}.$

Theorem 2 For any linear transformations $L: V \to W$ and $T: W \to X$,

 $[T \circ L]^{\gamma}_{\alpha} = [T]^{\gamma}_{\beta} [L]^{\beta}_{\alpha}.$

Theorem 3 For any linear operators $L: V \to V$ and $T: V \to V$, $[T \circ L]_{\alpha} = [T]_{\alpha}[L]_{\alpha}.$

Identity matrix

Definition. The **identity matrix** (or **unit matrix**) is an $n \times n$ matrix $I = (a_{ij})$ such that $a_{ii} = 1$ and $a_{ij} = 0$ for $i \neq j$. It is also denoted I_n .

$$I_{1} = (1), \quad I_{2} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad I_{3} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

n general,
$$I = \begin{pmatrix} 1 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 1 \end{pmatrix}.$$

Theorem. Let A be an arbitrary $m \times n$ matrix. Then $I_m A = A I_n = A$.

Inverse matrix

Definition. Let $A \in \mathcal{M}_{n,n}(\mathbb{F})$. Suppose there exists an $n \times n$ matrix B such that $AB = BA = I_n$. Then the matrix A is called **invertible** and B is called the **inverse** of A (denoted A^{-1}).

$$AA^{-1} = A^{-1}A = I$$

Basic properties of inverse matrices:

• If $B = A^{-1}$ then $A = B^{-1}$. In other words, if A is invertible, so is A^{-1} , and $A = (A^{-1})^{-1}$.

• The inverse matrix (if it exists) is unique.

• If $n \times n$ matrices A and B are invertible, so is AB, and $(AB)^{-1} = B^{-1}A^{-1}$.

• Similarly,
$$(A_1A_2...A_k)^{-1} = A_k^{-1}...A_2^{-1}A_1^{-1}$$
.

Inverting 2×2 matrices

Definition. The **determinant** of a 2×2 matrix $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is det A = ad - bc.

Theorem A matrix $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is invertible if and only if det $A \neq 0$.

If det $A \neq 0$ then $\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} = \frac{1}{ad - bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}.$ **Theorem** A matrix $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is invertible if and only if det $A \neq 0$. If det $A \neq 0$ then $\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} = \frac{1}{ad - bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}.$ *Proof:* Let $B = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$. Then $AB = BA = \begin{pmatrix} ad-bc & 0\\ 0 & ad-bc \end{pmatrix} = (ad-bc)I_2.$

In the case det $A \neq 0$, we have $A^{-1} = (\det A)^{-1}B$. In the case det A = 0, the matrix A is not invertible as otherwise $AB = O \implies A^{-1}(AB) = A^{-1}O = O$ $\implies (A^{-1}A)B = O \implies I_2B = O \implies B = O$ $\implies A = O$, but the zero matrix is not invertible.

Left multiplication

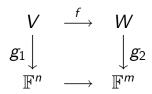
Any $m \times n$ matrix $A \in \mathcal{M}_{m,n}(\mathbb{F})$ gives rise to a linear transformation $L_A : \mathbb{F}^n \to \mathbb{F}^m$ given by $L_A(\mathbf{x}) = A\mathbf{x}$, where $\mathbf{x} \in \mathbb{F}^n$ and $L(\mathbf{x}) \in \mathbb{F}^m$ are regarded as column vectors.

Theorem 1 The matrix of the transformation L_A relative to the standard bases in \mathbb{F}^n and \mathbb{F}^m is exactly A.

Theorem 2 Suppose $L : \mathbb{F}^n \to \mathbb{F}^m$ is a linear map. Then there exists an $m \times n$ matrix A such that $L(\mathbf{x}) = A\mathbf{x}$ for all $\mathbf{x} \in \mathbb{F}^n$. Columns of A are vectors $L(\mathbf{e}_1), L(\mathbf{e}_2), \ldots, L(\mathbf{e}_n)$, where $\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n$ is the standard basis for \mathbb{F}^n .

Matrix of a linear transformation (revisited)

Let V, W be vector spaces and $f: V \to W$ be a linear map. Let $\alpha = [\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n]$ be a basis for V and $g_1: V \to \mathbb{F}^n$ be the coordinate mapping corresponding to this basis. Let $\beta = [\mathbf{w}_1, \dots, \mathbf{w}_m]$ be a basis for W and $g_2: W \to \mathbb{F}^m$ be the coordinate mapping corresponding to this basis.



The composition $g_2 \circ f \circ g_1^{-1}$ is a linear mapping of \mathbb{F}^n to \mathbb{F}^m . It is uniquely represented as $\mathbf{x} \mapsto A\mathbf{x}$, where $A \in \mathcal{M}_{m,n}(\mathbb{F})$.

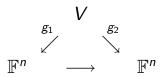
Theorem $A = [f]_{\alpha}^{\beta}$, the matrix of the transformation f relative to the bases α and β .

Change of coordinates

Let V be a vector space of dimension n.

Let $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n$ be a basis for V and $g_1 : V \to \mathbb{F}^n$ be the coordinate mapping corresponding to this basis.

Let $\mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_n$ be another basis for V and $g_2 : V \to \mathbb{F}^n$ be the coordinate mapping corresponding to this basis.



The composition $g_2 \circ g_1^{-1}$ is a linear operator on \mathbb{F}^n . It has the form $\mathbf{x} \mapsto U\mathbf{x}$, where U is an $n \times n$ matrix. U is called the **transition matrix** from $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$ to $\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_n$. Columns of U are coordinates of the vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$ with respect to the basis $\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_n$. **Problem.** Find the transition matrix from the basis $\mathbf{v}_1 = (1, 2, 3)$, $\mathbf{v}_2 = (1, 0, 1)$, $\mathbf{v}_3 = (1, 2, 1)$ to the basis $\mathbf{u}_1 = (1, 1, 0)$, $\mathbf{u}_2 = (0, 1, 1)$, $\mathbf{u}_3 = (1, 1, 1)$.

It is convenient to make a two-step transition: first from $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ to $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$, and then from $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$ to $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3$.

Let U_1 be the transition matrix from $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ to $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$ and U_2 be the transition matrix from $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3$ to $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$:

$$U_1 = \begin{pmatrix} 1 & 1 & 1 \\ 2 & 0 & 2 \\ 3 & 1 & 1 \end{pmatrix}, \qquad U_2 = \begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{pmatrix}$$

Basis $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3 \implies$ coordinates \mathbf{x} Basis $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3 \implies$ coordinates $U_1 \mathbf{x}$ Basis $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3 \implies$ coordinates $U_2^{-1}(U_1 \mathbf{x}) = (U_2^{-1}U_1) \mathbf{x}$

Thus the transition matrix from $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ to $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3$ is $U_2^{-1}U_1$.

$$U_2^{-1}U_1 = \begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{pmatrix}^{-1} \begin{pmatrix} 1 & 1 & 1 \\ 2 & 0 & 2 \\ 3 & 1 & 1 \end{pmatrix}$$
$$= \begin{pmatrix} 0 & 1 & -1 \\ -1 & 1 & 0 \\ 1 & -1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 & 1 \\ 2 & 0 & 2 \\ 3 & 1 & 1 \end{pmatrix} = \begin{pmatrix} -1 & -1 & 1 \\ 1 & -1 & 1 \\ 2 & 2 & 0 \end{pmatrix}$$

Problem. Consider a linear operator $L : \mathbb{F}^2 \to \mathbb{F}^2$,

$$L\begin{pmatrix} x\\ y \end{pmatrix} = \begin{pmatrix} 1 & 1\\ 0 & 1 \end{pmatrix} \begin{pmatrix} x\\ y \end{pmatrix}.$$

Find the matrix of L with respect to the basis $\mathbf{v}_1 = (3, 1)$, $\mathbf{v}_2 = (2, 1)$.

Let *N* be the desired matrix. Columns of *N* are coordinates of the vectors $L(\mathbf{v}_1)$ and $L(\mathbf{v}_2)$ w.r.t. the basis $\mathbf{v}_1, \mathbf{v}_2$.

$$\begin{split} \mathcal{L}(\mathbf{v}_1) &= \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 3 \\ 1 \end{pmatrix} = \begin{pmatrix} 4 \\ 1 \end{pmatrix}, \quad \mathcal{L}(\mathbf{v}_2) = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 3 \\ 1 \end{pmatrix}. \\ \text{Clearly, } \mathcal{L}(\mathbf{v}_2) &= \mathbf{v}_1 = 1\mathbf{v}_1 + 0\mathbf{v}_2. \end{split}$$

$$L(\mathbf{v}_1) = a\mathbf{v}_1 + b\mathbf{v}_2 \iff \begin{cases} 3a + 2b = 4\\ a + b = 1 \end{cases} \iff \begin{cases} a = 2\\ b = -1 \end{cases}$$

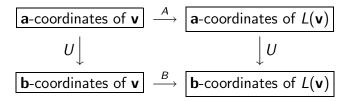
Thus $N = \begin{pmatrix} 2 & 1\\ -1 & 0 \end{pmatrix}$.

Change of coordinates for a linear operator

Let $L: V \rightarrow V$ be a linear operator on a vector space V.

Let A be the matrix of L relative to a basis $\mathbf{a}_1, \mathbf{a}_2, \dots, \mathbf{a}_n$ for V. Let B be the matrix of L relative to another basis $\mathbf{b}_1, \mathbf{b}_2, \dots, \mathbf{b}_n$ for V.

Let U be the transition matrix from the basis $\mathbf{a}_1, \mathbf{a}_2, \dots, \mathbf{a}_n$ to $\mathbf{b}_1, \mathbf{b}_2, \dots, \mathbf{b}_n$.



It follows that $UA\mathbf{x} = BU\mathbf{x}$ for all $\mathbf{x} \in \mathbb{F}^n \implies UA = BU$. Then $A = U^{-1}BU$ and $B = UAU^{-1}$. **Problem.** Consider a linear operator $L : \mathbb{F}^2 \to \mathbb{F}^2$, $L \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}.$

Find the matrix of L with respect to the basis $\mathbf{v}_1 = (3, 1)$, $\mathbf{v}_2 = (2, 1)$.

Let *S* be the matrix of *L* with respect to the standard basis, *N* be the matrix of *L* with respect to the basis \mathbf{v}_1 , \mathbf{v}_2 , and *U* be the transition matrix from \mathbf{v}_1 , \mathbf{v}_2 to \mathbf{e}_1 , \mathbf{e}_2 . Then $N = U^{-1}SU$.

$$S = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \quad U = \begin{pmatrix} 3 & 2 \\ 1 & 1 \end{pmatrix},$$
$$N = U^{-1}SU = \begin{pmatrix} 1 & -2 \\ -1 & 3 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 3 & 2 \\ 1 & 1 \end{pmatrix}$$
$$= \begin{pmatrix} 1 & -1 \\ -1 & 2 \end{pmatrix} \begin{pmatrix} 3 & 2 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 2 & 1 \\ -1 & 0 \end{pmatrix}.$$