MATH 423 Linear Algebra II

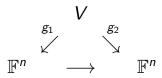
Lecture 11: Change of coordinates (continued). Isomorphism of vector spaces.

Change of coordinates

Let V be a vector space of dimension n.

Let $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n$ be a basis for V and $g_1 : V \to \mathbb{F}^n$ be the coordinate mapping corresponding to this basis.

Let $\mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_n$ be another basis for V and $g_2: V \to \mathbb{F}^n$ be the coordinate mapping corresponding to this basis.



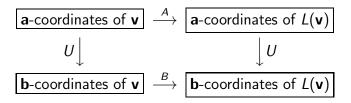
The composition $g_2 \circ g_1^{-1}$ is a linear operator on \mathbb{F}^n . It has the form $\mathbf{x} \mapsto U\mathbf{x}$, where U is an $n \times n$ matrix. U is called the **transition matrix** from $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$ to $\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_n$. Columns of U are coordinates of the vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$ with respect to the basis $\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_n$.

Change of coordinates for a linear operator

Let $L: V \to V$ be a linear operator on a vector space V.

Let A be the matrix of L relative to a basis $\mathbf{a}_1, \mathbf{a}_2, \dots, \mathbf{a}_n$ for V. Let B be the matrix of L relative to another basis $\mathbf{b}_1, \mathbf{b}_2, \dots, \mathbf{b}_n$ for V.

Let U be the transition matrix from the basis $\mathbf{a}_1, \mathbf{a}_2, \dots, \mathbf{a}_n$ to $\mathbf{b}_1, \mathbf{b}_2, \dots, \mathbf{b}_n$.



It follows that $UA\mathbf{x} = BU\mathbf{x}$ for all $\mathbf{x} \in \mathbb{F}^n \implies UA = BU$. Then $A = U^{-1}BU$ and $B = UAU^{-1}$. **Problem.** Consider a linear operator $L : \mathbb{F}^2 \to \mathbb{F}^2$, $L \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$.

Find the matrix of L with respect to the basis $\mathbf{v}_1 = (3, 1)$, $\mathbf{v}_2 = (2, 1)$.

Let *S* be the matrix of *L* with respect to the standard basis, *N* be the matrix of *L* with respect to the basis \mathbf{v}_1 , \mathbf{v}_2 , and *U* be the transition matrix from \mathbf{v}_1 , \mathbf{v}_2 to \mathbf{e}_1 , \mathbf{e}_2 . Then $N = U^{-1}SU$.

$$S = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \quad U = \begin{pmatrix} 3 & 2 \\ 1 & 1 \end{pmatrix},$$
$$N = U^{-1}SU = \begin{pmatrix} 1 & -2 \\ -1 & 3 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 3 & 2 \\ 1 & 1 \end{pmatrix}$$
$$= \begin{pmatrix} 1 & -1 \\ -1 & 2 \end{pmatrix} \begin{pmatrix} 3 & 2 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 2 & 1 \\ -1 & 0 \end{pmatrix}.$$

Similarity

Definition. An $n \times n$ matrix B is said to be similar to an $n \times n$ matrix A if $B = S^{-1}AS$ for some nonsingular $n \times n$ matrix S.

Remark. Two $n \times n$ matrices are similar if and only if they represent the same linear operator on \mathbb{F}^n with respect to different bases.

Theorem Similarity is an *equivalence relation*, which means that

(i) any square matrix A is similar to itself;
(ii) if B is similar to A, then A is similar to B;
(iii) if A is similar to B and B is similar to C, then A is similar to C.

Theorem Let V, W be finite-dimensional vector spaces and $f: V \rightarrow W$ be a linear map. Then one can choose bases for V and W so that the respective matrix of f is has the block form

 $\begin{pmatrix} I_r & O \\ O & O \end{pmatrix},$

where r is the rank of f.

Example. With a suitable choice of bases, any linear map $f : \mathbb{F}^3 \to \mathbb{F}^2$ has one of the following matrices:

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}.$$

Proof of the theorem:

Let $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_k$ be a basis for the null-space $\mathcal{N}(f)$.

Extend it to a basis $\mathbf{v}_1, \ldots, \mathbf{v}_k, \mathbf{u}_1, \ldots, \mathbf{u}_r$ for V.

Then $f(\mathbf{u}_1), f(\mathbf{u}_2), \dots, f(\mathbf{u}_r)$ is a basis for the range $\mathcal{R}(f)$.

Extend it to a basis $f(\mathbf{u}_1), \ldots, f(\mathbf{u}_r), \mathbf{w}_1, \ldots, \mathbf{w}_l$ for W.

Now the matrix of f with respect to bases $[\mathbf{u}_1, \ldots, \mathbf{u}_r, \mathbf{v}_1, \ldots, \mathbf{v}_k]$ and $[f(\mathbf{u}_1), \ldots, f(\mathbf{u}_r), \mathbf{w}_1, \ldots, \mathbf{w}_l]$ is $\begin{pmatrix} I_r & O \\ O & O \end{pmatrix}$.

Definition. A map $f: V_1 \rightarrow V_2$ is **one-to-one** if it maps different elements from V_1 to different elements in V_2 . The map f is **onto** if any element $\mathbf{y} \in V_2$ is represented as $f(\mathbf{x})$ for some $\mathbf{x} \in V_1$.

If the map f is both one-to-one and onto, then the inverse map $f^{-1}: V_2 \to V_1$ is well defined.

Now let V_1, V_2 be vector spaces and $L: V_1 \rightarrow V_2$ be a linear map.

Theorem (i) The linear map L is one-to-one if and only if $\mathcal{N}(L) = \{\mathbf{0}\}$.

(ii) The linear map L is onto if $\mathcal{R}(L) = V_2$. (iii) If the linear map L is both one-to-one and onto, then the inverse map L^{-1} is also linear.

Isomorphism

Definition. A linear map $L: V_1 \rightarrow V_2$ is called an **isomorphism** of vector spaces if it is both one-to-one and onto.

The vector space V_1 is said to be **isomorphic** to V_2 if there exists an isomorphism $L: V_1 \rightarrow V_2$.

The word "isomorphism" applies when two complex structures can be mapped onto each other, in such a way that to each part of one structure there is a corresponding part in the other structure, where "corresponding" means that the two parts play similar roles in their respective structures.

Alternative notation

General maps

one-to-one
ontosurjective
one-to-one and onto bijective
Linear maps
any maphomomorphism
one-to-onemonomorphism
onto
one-to-one and onto isomorphism
Linear self-maps
any map endomorphism
one-to-one and onto automorphism

Examples of isomorphism

• $\mathcal{M}_{1,3}(\mathbb{F})$ is isomorphic to $\mathcal{M}_{3,1}(\mathbb{F})$. Isomorphism: $(x_1, x_2, x_3) \mapsto \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$.

•
$$\mathcal{M}_{2,2}(\mathbb{F})$$
 is isomorphic to \mathbb{F}^4 .
Isomorphism: $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto (a, b, c, d)$.

• $\mathcal{M}_{2,3}(\mathbb{F})$ is isomorphic to $\mathcal{M}_{3,2}(\mathbb{F})$. Isomorphism: $A \mapsto A^t$.

• The plane z = 0 in \mathbb{R}^3 is isomorphic to \mathbb{R}^2 . Isomorphism: $(x, y, 0) \mapsto (x, y)$.

Examples of isomorphism

• \mathcal{P}_n is isomorphic to \mathbb{R}^{n+1} . Isomorphism: $a_0+a_1x+\cdots+a_nx^n\mapsto (a_0,a_1,\ldots,a_n)$.

• \mathcal{P} is isomorphic to \mathbb{R}_0^{∞} . Isomorphism:

 $a_0 + a_1x + \cdots + a_nx^n \mapsto (a_0, a_1, \ldots, a_n, 0, 0, \ldots).$

• $\mathcal{M}_{m,n}(\mathbb{F})$ is isomorphic to $\mathcal{L}(\mathbb{F}^n, \mathbb{F}^m)$. Isomorphism: $A \mapsto L_A$, where $L_A(\mathbf{x}) = A\mathbf{x}$.

• Any vector space V of dimension n is isomorphic to \mathbb{F}^n .

Isomorphism: $\mathbf{v} \mapsto [\mathbf{v}]_{\alpha}$, where α is a basis for V.

Isomorphism and dimension

Definition. Two sets S_1 and S_2 are said to be of the same **cardinality** if there exists a bijective map $f: S_1 \rightarrow S_2$.

Theorem 1 All bases of a fixed vector space V are of the same cardinality.

Theorem 2 Two vector spaces are isomorphic if and only if their bases are of the same cardinality. In particular, a vector space V is isomorphic to \mathbb{F}^n if and only if dim V = n.

Remark. For a finite set, the cardinality is a synonym for the number of its elements. For an infinite set, the cardinality is a more sophisticated notion. For example, \mathbb{R}^{∞} and \mathcal{P} are both infinite-dimensional vector spaces but they are not isomorphic.