MATH 433

Applied Algebra

Lecture 16:
Algebraic structures (continued).

Ring

Definition. A **ring** is a set R, together with two binary operations usually called **addition** and **multiplication** and denoted accordingly, such that

- R is an Abelian group under addition,
- R is a semigroup under multiplication,
- multiplication distributes over addition.

A ring R is called **commutative** if the multiplication is commutative. R is called a **ring with identity** if there exists an identity element for multiplication (denoted 1).

An **integral domain** is a nontrivial commutative ring with identity and no zero-divisors (i.e., $ab = 0$ implies $a = 0$ or $b = 0$).

Examples of rings

- Real numbers $\mathbb{R}$.
- Integers $\mathbb{Z}$.
- $2\mathbb{Z}$: even integers.
- $\mathbb{Z}_n$: congruence classes modulo n.
- $\mathcal{M}_n(\mathbb{R})$: all $n \times n$ matrices with real entries.
- $\mathcal{M}_n(\mathbb{Z})$: all $n \times n$ matrices with integer entries.
- $\mathcal{M}_n(R)$: all $n \times n$ matrices with entries from a ring R.
- $\mathbb{R}[X]$: polynomials in variable X with real coefficients.
- $\mathbb{Z}[X]$: polynomials in variable X with integer coefficients.
- $R[X]$: polynomials in variable X with coefficients from a ring R.
- $\mathbb{R}(X)$: rational functions in variable X with real coefficients.
- All functions $f : \mathbb{R} \to \mathbb{R}$.

Field

Definition. A **field** is a set F, together with two binary operations called **addition** and **multiplication** and denoted accordingly, such that

- F is an Abelian group under addition,
- $F \setminus \{0\}$ is an Abelian group under multiplication,
- multiplication distributes over addition.

In other words, the field is an integral domain such that any nonzero element has a multiplicative inverse.

Examples. • Real numbers $\mathbb{R}$.

• Rational numbers $\mathbb{Q}$.

• $\mathbb{Z}_p$: congruence classes modulo p, where p is prime.

• $\mathbb{R}(X)$: rational functions in variable X with real coefficients.

• $F(X)$: rational functions in variable X with coefficients from a field F.

Quadratic extension

Consider the set $\mathbb{Z}[\sqrt{2}]$ of all numbers of the form $a + b\sqrt{2}$, where $a, b \in \mathbb{Z}$. This set is closed under addition, subtraction, and multiplication:

$$(a + b\sqrt{2}) + (c + d\sqrt{2}) = (a + c) + (b + d)\sqrt{2},$$
$$(a + b\sqrt{2}) - (c + d\sqrt{2}) = (a - c) + (b - d)\sqrt{2},$$
$$(a + b\sqrt{2})(c + d\sqrt{2}) = (ac + 2bd) + (ad + bc)\sqrt{2}.$$

It follows that $\mathbb{Z}[\sqrt{2}]$ is a ring. Actually, it is an integral domain. The quotient field of $\mathbb{Z}[\sqrt{2}]$ is $\mathbb{Q}(\sqrt{2})$, the set of all fractions $\frac{a+b\sqrt{2}}{c+d\sqrt{2}}$, where $a, b, c, d \in \mathbb{Q}$ and $|c| + |d| \neq 0$.
In fact, $\mathbb{Q}(\sqrt{2}) = \mathbb{Q}[\sqrt{2}]$:

$$\frac{1}{c + d\sqrt{2}} = \frac{c - d\sqrt{2}}{(c + d\sqrt{2})(c - d\sqrt{2})} = \frac{c}{c^2 - 2d^2} - \frac{d}{c^2 - 2d^2}\sqrt{2}.$$

The field $\mathbb{Q}[\sqrt{2}]$ is a **quadratic extension** of the field $\mathbb{Q}$.
Similarly, the field $\mathbb{C}$ is a quadratic extension of $\mathbb{R}$, $\mathbb{C} = \mathbb{R}[\sqrt{-1}]$.

Vector space over a field

Definition. Given a field F, a **vector space** V over F is an additive Abelian group endowed with an action of F called **scalar multiplication** or **scaling**.

An **action** of F on V is an operation that takes elements $\lambda \in F$ and $v \in V$ and gives an element, denoted λv, of V.

The scalar multiplication is to satisfy the following axioms:

(V1) for all $v \in V$ and $\lambda \in F$, λv is an element of V;

(V2) $\lambda(\mu v) = (\lambda\mu)v$ for all $v \in V$ and $\lambda, \mu \in F$;

(V3) $1v = v$ for all $v \in V$;

(V4) $(\lambda + \mu)v = \lambda v + \mu v$ for all $v \in V$ and $\lambda, \mu \in F$;

(V5) $\lambda(v + w) = \lambda v + \lambda w$ for all $v, w \in V$ and $\lambda \in F$.

Examples of vector spaces over a field F:

- The space F^n of n-dimensional coordinate vectors $(x_1, x_2, \ldots, x_n)$ with coordinates in F.
- The space $\mathcal{M}_{n,m}(F)$ of $n \times m$ matrices with entries in F.
- The space $F[X]$ of polynomials $p(x) = a_0 + a_1 X + \cdots + a_n X^n$ with coefficients in F.
- Any field F' that is an extension of F (i.e., $F \subset F'$ and the operations on F are restrictions of the corresponding operations on F'). In particular, $\mathbb{C}$ is a vector space over $\mathbb{R}$ and over $\mathbb{Q}$, $\mathbb{R}$ is a vector space over $\mathbb{Q}$, $\mathbb{Q}[\sqrt{2}]$ is a vector space over $\mathbb{Q}$.

Characteristic of a field

A field F is said to be of nonzero characteristic if $\underbrace{1+1+\cdots+1}_{n \text{ times}}=0$ for some positive integer n. The smallest integer with this property is the **characteristic** of F. Otherwise the field F has characteristic 0.

The fields $\mathbb{Q}$, $\mathbb{R}$, $\mathbb{C}$ have characteristic 0. The field $\mathbb{Z}_p$ (p prime) has characteristic p.

Since $(\underbrace{1+\cdots+1}_{n \text{ times}})(\underbrace{1+\cdots+1}_{m \text{ times}})=\underbrace{1+\cdots+1}_{nm \text{ times}}$, any nonzero characteristic is prime.

Any field of characteristic 0 has a unique structure of the vector space over $\mathbb{Q}$. Any field of characteristic $p>0$ has a unique structure of the vector space over $\mathbb{Z}_p$. It follows that any finite field F of charasteristic p has p^n elements (where n is the dimension of F as a vector space over $\mathbb{Z}_p$).

Algebra over a field

Definition. An **algebra** A over a field F (or F-**algebra**) is a vector space with a multiplication which is a bilinear operation on A. That is, the product xy is both a linear function of x and a linear function of y.

To be precise, the following axioms are to be satisfied:

(A1) for all $x, y \in A$, the product xy is an element of A;
(A2) $x(y+z) = xy+xz$ and $(y+z)x = yx+zx$ for $x, y, z \in A$;
(A3) $(\lambda x)y = \lambda(xy) = x(\lambda y)$ for all $x, y \in A$ and $\lambda \in F$.

An F-algebra is **associative** if the multiplication is associative. An associative algebra is both a vector space and a ring.

An F-algebra A is a **Lie algebra** if the multiplication (usually denoted $[x, y]$ in this case) satisfies the following conditions:

(Antisymmetry): $[x, y] = -[y, x]$ for all $x, y \in A$;
(Jacobi's identity): $[[x, y], z] + [[y, z], x] + [[z, x], y] = 0$ for all $x, y, z \in A$.

Examples of associative algebras:

- The space $\mathcal{M}_n(F)$ of $n \times n$ matrices with entries in F.
- The space $F[X]$ of polynomials $p(x) = a_0 + a_1 X + \cdots + a_n X^n$ with coefficients in F.
- The space of all functions $f : S \to F$ on a set S taking values in a field F.
- Any field F' that is an extension of a field F is an associative algebra over F.

Examples of Lie algebras:

- $\mathbb{R}^3$ with the cross product is a Lie algebra over $\mathbb{R}$.
- Any associative algebra A with an alternative multiplication defined by $[x, y] = xy - yx$.

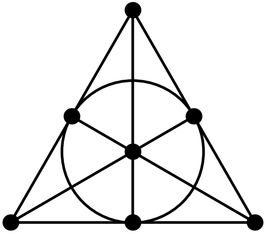
Finite projective plane

A **projective plane** is a set P of points, together with selected subsets called **lines**, such that **(i)** there is exactly one line containing any two distinct points, **(ii)** any two distinct lines intersect at a single point, and **(iii)** there are 4 points no 3 of which lie on the same line.

A **projective transformation** of the plane P is a bijection $f : P \to P$ that sends lines to lines. All projective transformations of P form a transformation group.

The smallest projective plane (called the **Fano plane**) has 7 points. It also has 7 lines, each line consisting of 3 points.

Fano plane



The Fano plane can be realized as the set of nonzero vectors in $\mathbb{Z}_2^3$, a 3-dimensional vector space over the field $\mathbb{Z}_2$. Each line has the form $\ell \setminus \{(0,0,0)\}$, where ℓ is a 2-dimensional subspace of $\mathbb{Z}_2^3$.

In this realization, the projective transformations of the Fano plane correspond to invertible linear operators on $\mathbb{Z}_2^3$. Hence the group of all projective transformations can be identified with the group $GL(3,\mathbb{Z}_2)$ of 3×3 matrices with entries from $\mathbb{Z}_2$ and nonzero determinant. This group has 168 elements.