MATH 614

Dynamical Systems and Chaos

# Lecture 3: Classification of fixed points. Logistic map.

# Periodic points

*Definition.* A point $x \in X$ is called a **fixed** point of a map $f : X \to X$ if $f(x) = x$.

A point $x \in X$ is called a **periodic** point of a map $f : X \to X$ if $f^m(x) = x$ for some integer $m \geq 1$. The least integer $m$ satisfying this relation is called the **period** of $x$.

A point $x \in X$ is called an **eventually periodic** point of the map $f$ if for some integer $k \geq 0$ the point $f^k(x)$ is a periodic point of $f$.

# Properties of periodic points

- If $x$ is a periodic point of period $m$, then $f^n(x) = x$ if and only if $m$ divides $n$.
- If $x$ is a periodic point of period $m$, then $f^{n_1}(x) = f^{n_2}(x)$ if and only if $m$ divides $n_1 - n_2$.
- If $x$ is a periodic point of period $m$, then the orbit of $x$ consists of $m$ points.
- If $x$ is a periodic point, then every element of the orbit of $x$ is also a periodic point of the same period.
- A point is eventually periodic if and only if its orbit is finite (as a set).
- If the map $f$ is invertible, then every eventually periodic point is actually periodic.

# Classification of fixed points

Let $X$ be a subset of $\mathbb{R}$, $f : X \to X$ be a continuous map, and $x_0$ be a fixed point of $f$.

*Definition.* The **stable set** of the fixed point $x_0$, denoted $W^s(x_0)$, consists of all points $x \in X$ such that $f^n(x) \to x_0$ as $n \to \infty$. In the case $f$ is invertible, the **unstable set** of $x_0$, denoted $W^u(x_0)$, is the stable set of $x_0$ considered as a fixed point of $f^{-1}$.

The fixed point $x_0$ is **weakly attracting** if the stable set $W^s(x_0)$ contains an open neighborhood of $x_0$, i.e., $(x_0 - \varepsilon, x_0 + \varepsilon) \subset W^s(x_0)$ for some $\varepsilon > 0$.

The fixed point $x_0$ is **weakly repelling** if there exists an open neighborhood $U$ of $x_0$ such that for each $x \in U \setminus \{x_0\}$ the orbit $O^+(x)$ is not completely contained in $U$.

**Proposition** Suppose $f$ is invertible. Then a fixed point $x_0$ of $f$ is weakly repelling if and only if $x_0$ is a weakly attracting fixed point of the inverse map $f^{-1}$.

*Proof:* It is no loss to assume that the domain of $f$ includes an interval $U = (x_0 - \varepsilon, x_0 + \varepsilon)$ for some $\varepsilon > 0$. Since the map $f$ is continuous and invertible, it is strictly monotone on $U$. Let us choose $\varepsilon_0$, $0 < \varepsilon_0 < \varepsilon$, so that $f(U_0) \subset U$, where $U_0 = (x_0 - \varepsilon_0, x_0 + \varepsilon_0)$. Then $f^2$ is strictly increasing on $U_0$.

If $x_0$ is not an isolated fixed point of $f^2$, then it is neither weakly attracting nor weakly repelling for both $f$ and $f^{-1}$. Therefore it is no loss to assume that $x_0$ is the only fixed point of $f^2$ in the interval $U_0$. Then the function $g(x) = f^2(x) - x$ maintains its sign on $(x_0 - \varepsilon_0, x_0)$ and on $(x_0, x_0 + \varepsilon_0)$. If those signs are $-$ and $+$, then $x_0$ is both weakly repelling for $f^2$ and weakly attracting for $f^{-2}$. Otherwise $x_0$ is neither.

Finally, $x_0$ is weakly repelling for $f$ if and only if it is so for $f^2$. Also, $x_0$ is weakly attracting for $f^{-1}$ if and only if it is for $f^{-2}$.

# Classification of fixed points (continued)

*Definition.* The fixed point $x_0$ is **attracting** if for some $\lambda \in (0,1)$ there exists an open interval $U$ containing $x_0$ such that $|f(x) - x_0| \le \lambda |x - x_0|$ for all $x \in U$. The point $x_0$ is **super-attracting** if such an interval exists for any $\lambda \in (0,1)$.

It is no loss to assume that $U = (x_0 - \varepsilon, x_0 + \varepsilon)$ for some $\varepsilon > 0$. Then it follows that $f(U) \subset U$ and $|f^n(x) - x_0| \le \lambda^n |x - x_0|$ for all $x \in U$ and $n = 1, 2, \dots$ In particular, the orbit of any point $x \in U$ converges to $x_0$. Hence an attracting fixed point is weakly attracting as well.

*Definition.* The fixed point $x_0$ is **repelling** if there exist $\lambda > 1$ and an open interval $U$ containing $x_0$ such that $|f(x) - x_0| \ge \lambda |x - x_0|$ for all $x \in U$.

It is easy to observe that any repelling fixed point is weakly repelling as well.

**Theorem** Suppose that a map $f : X \to X$ is differentiable at a fixed point $x_0$ and let $\lambda = f'(x_0)$ be the multiplier of $x_0$. Then **(i)** $x_0$ is attracting if and only if $|\lambda| < 1$;
**(ii)** $x_0$ is super-attracting if and only if $\lambda = 0$;
**(iii)** $x_0$ is repelling if and only if $|\lambda| > 1$.

*Proof:* Since $\lambda = f'(x_0)$, for any $\delta > 0$ there exists $\varepsilon > 0$ such that

$$\lambda - \delta < \frac{f(x) - x_0}{x - x_0} < \lambda + \delta \quad \text{whenever} \quad 0 < |x - x_0| < \varepsilon.$$

Then $(|\lambda| - \delta)|x - x_0| \leq |f(x) - x_0| \leq (|\lambda| + \delta)|x - x_0|$ for $|x - x_0| < \varepsilon$. Notice that the numbers $|\lambda| - \delta$ and $|\lambda| + \delta$ can be made arbitrarily close to $|\lambda|$.

In the case $|\lambda| < 1$, we obtain that $x_0$ is an attracting fixed point. Furthermore, if $\lambda = 0$, then $x_0$ is super-attracting. However $x_0$ is not super-attracting if $\lambda \neq 0$. In the case $|\lambda| > 1$, we obtain that $x_0$ is a repelling fixed point. Finally, in the case $|\lambda| = 1$ the fixed point $x_0$ is neither attracting nor repelling.

# Classification of periodic points

Let $X$ be a subset of $\mathbb{R}$, $f : X \to X$ be a continuous map, and $x_0$ be a periodic point of $f$ with period $m$. Then $x_0$ is a fixed point of the map $f^m$.

The **stable set** of the periodic point $x_0$, denoted $W^s(x_0)$, is defined as the stable set of the same point considered a fixed point of $f^m$. That is, $W^s(x_0)$ consists of all points $x \in X$ such that $f^{nm}(x) \to x_0$ as $n \to \infty$.

In the case $f$ is invertible, so is the map $f^m$. In this case the **unstable set** of $x_0$, denoted $W^u(x_0)$, is defined as the stable set of $x_0$ considered a fixed point of $(f^m)^{-1}$.

The periodic point $x_0$ is called **weakly attracting** (resp. **attracting**, **super-attracting**, **weakly repelling**, **repelling**) if it enjoys the same property as a fixed point of $f^m$.

# Newton's method

**Newton's method** is an iterative process for finding roots of a polynomial. Given a nonconstant polynomial $Q$, consider a rational function

$$f(x) = x - \frac{Q(x)}{Q'(x)}.$$

It turns out that, for a properly chosen initial point $x_0$, the orbit $x_0, f(x_0), f^2(x_0), \dots$ converges very fast to a root of $Q$.

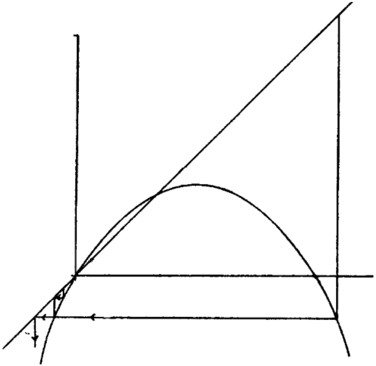
Suppose $z$ is a simple root of $Q$, that is, $Q(z) = 0$ while $Q'(z) \neq 0$. Clearly, $z$ is a fixed point of the map $f$. We have

$$f'(z) = 1 - \frac{Q'(z)Q'(z) - Q(z)Q''(z)}{(Q'(z))^2} = \frac{Q(z)Q''(z)}{(Q'(z))^2} = 0.$$

Thus $z$ is a super-attracting fixed point of $f$.

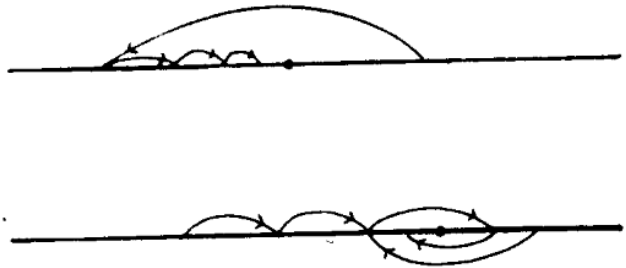
# Logistic map

The **logistic map** is any of the family of quadratic maps $F_\mu(x) = \mu x(1-x)$ depending on the parameter $\mu \in \mathbb{R}$.
In the case $\mu > 1$, the map $F_\mu$ has two fixed points: 0 and $p_\mu = 1 - \mu^{-1} \in (0,1)$. Besides, 1 and $\mu^{-1}$ are eventually fixed points. Orbits of all points outside of the interval $[0,1]$ diverge to $-\infty$.



# Logistic map: $1 < \mu < 3$

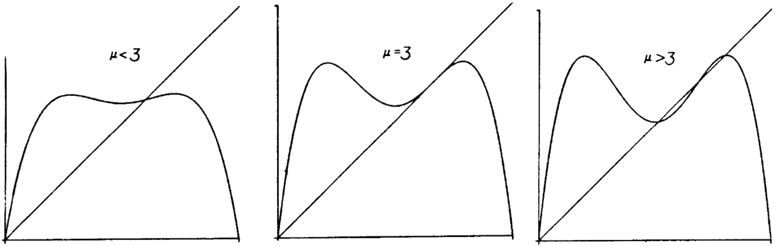
In the case $1 < \mu < 3$, the fixed point $p_\mu = (\mu - 1)/\mu$ is attracting. Moreover, the orbit of any point $x \in (0, 1)$ converges to $p_\mu$.



These are phase portraits of $F_\mu$ near the fixed point $p_\mu$ for $1 < \mu < 2$ and $2 < \mu < 3$.

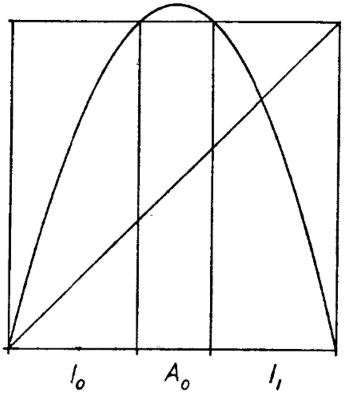
# Logistic map: $\mu \approx 3$

The graphs of $F_\mu^2$ for $\mu \approx 3$:




For $\mu < 3$, the fixed point $p_\mu$ is attracting. At $\mu = 3$, it is not hyperbolic. For $\mu > 3$, the fixed point $p_\mu$ is repelling and there is also an attracting periodic orbit of period 2.

# Logistic map: $\mu > 4$




For $\mu > 4$, the interval $I = [0, 1]$ is no longer invariant under the map $F_\mu$. Instead, it is now split into 3 subintervals: $I = I_0 \cup A_0 \cup I_1$, where $I_0$ and $I_1$ are mapped bijectively onto $I$ while $A_0$ is mapped to the complement of $I$.