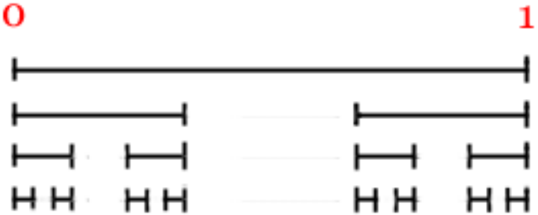
MATH 614

Dynamical Systems and Chaos

Lecture 5:
Cantor sets (continued).
Metric and topological spaces.
Symbolic dynamics.

Cantor Middle-Thirds Set



General Cantor sets

Definition. A subset Λ of the real line $\mathbb{R}$ is called a (general) **Cantor set** if it is

- nonempty,
- compact, which means that Λ is bounded and closed,
- totally disconnected, which means that Λ contains no intervals, and
- perfect, which means that Λ has no isolated points.

Theorem Any two Cantor sets are homeomorphic. That is, if Λ and Λ' are Cantor sets, then there exists a homeomorphism $\phi : \Lambda \to \Lambda'$.

Furthermore, the homeomorphism ϕ can be chosen strictly increasing, in which case it can be extended to a homeomorphism $\tilde{\phi} : \mathbb{R} \to \mathbb{R}$.

An open subset $U \subset \mathbb{R}$ is a union of open intervals. An open interval (a, b) is called a **maximal subinterval** of U if there is no other interval (c, d) such that $(a, b) \subset (c, d) \subset U$.

Lemma 1 Any point of U is contained in a maximal subinterval.

Lemma 2 Finite endpoints of a maximal subinterval do not belong to U.

Lemma 3 Distinct maximal subintervals are disjoint.

Lemma 4 There are at most countably many maximal subintervals.

Lemma 5 If Λ is a Cantor set, then for any two maximal subintervals of $\mathbb{R} \setminus \Lambda$ there is another maximal subinterval that lies between them.

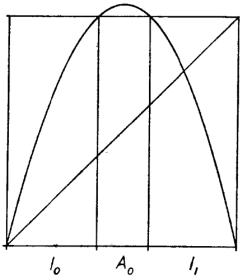
Lemma 6 If Λ, Λ' are Cantor sets sets then there exists a monotone one-to-one correspondence between maximal subintervals of their complements.

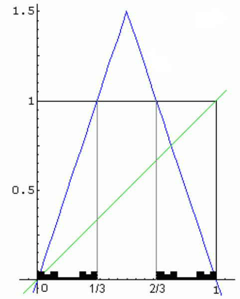
Unimodal maps

Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous map such that

- $f(0) = f(1) = 0$;
- there exists a point $x_{\max} \in (0,1)$ such that f is strictly increasing on $(-\infty, x_{\max}]$ and strictly decreasing on $[x_{\max}, \infty)$;
- $f(x_{\max}) > 1$.

The map f is called **unimodal**.





Itinerary map

Let $f : \mathbb{R} \to \mathbb{R}$ be a unimodal map, Λ be the set of all points $x \in \mathbb{R}$ such that $O_f^+(x) \subset [0, 1]$, and $S : \Lambda \to \Sigma_2$ be the **itinerary map** introduced in the previous lecture.

Proposition 1 The set Λ is compact and has no isolated points.

Proposition 2 $S \circ f = \sigma \circ S$ on Λ, where $\sigma : \Sigma_2 \to \Sigma_2$ is the shift map.

Proposition 3 The itinerary map S is onto.

Proposition 4 The set Λ is a Cantor set if and only if the itinerary map S is one-to-one.

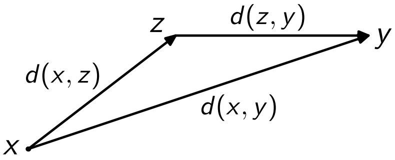
Metric space

Definition. Given a nonempty set X, a **metric** (or **distance function**) on X is a function $d : X \times X \to \mathbb{R}$ that satisfies the following conditions:

• **(positivity)** $d(x, y) \geq 0$ for all $x, y \in X$; moreover, $d(x, y) = 0$ if and only if $x = y$;

• **(symmetry)** $d(x, y) = d(y, x)$ for all $x, y \in X$;

• **(triangle inequality)** $d(x, y) \leq d(x, z) + d(z, y)$ for all $x, y, z \in X$.



A set endowed with a metric is called a **metric space**.

Examples of metric spaces

- *Real line*

$X = \mathbb{R}$, $d(x, y) = |y - x|$.

- *Euclidean space*

$X = \mathbb{R}^n$, $d(\mathbf{x}, \mathbf{y}) = \sqrt{(y_1 - x_1)^2 + (y_2 - x_2)^2 + \cdots + (y_n - x_n)^2}$.

- *Normed vector space*

X: vector space with a norm $\|\cdot\|$, $d(\mathbf{x}, \mathbf{y}) = \|\mathbf{y} - \mathbf{x}\|$.

- *Discrete metric space*

X: any nonempty set, $d(x, y) = 1$ if $x \neq y$ and $d(x, y) = 0$ if $x = y$.

- *Subspace of a metric space*

X: nonempty subset of a metric space Y with a distance function $\rho : Y \times Y \to \mathbb{R}$, d is the restriction of ρ to $X \times X$.

Convergence and continuity

Suppose (X, d) is a metric space, that is, X is a set and d is a metric on X.

We say that a sequence of points $x_1, x_2, \ldots$ of the set X **converges** to a point $y \in X$ if $d(x_n, y) \to 0$ as $n \to \infty$.

Given another metric space (Y, ρ) and a function $f : X \to Y$, we say that f is **continuous at a point** $x_0 \in X$ if for every $\varepsilon > 0$ there exists $\delta > 0$ such that $d(x, x_0) < \delta \implies \rho(f(x), f(x_0)) < \varepsilon$.

We say that the function f is **continuous on a set** $U \subset X$ if it is continuous at each point of U.

Open sets

Let (X, d) be a metric space. For any $x_0 \in X$ and $\varepsilon > 0$ we define the **open ball** (or simply **ball**) $B_\varepsilon(x_0)$ of radius ε centered at x_0 by $B_\varepsilon(x_0) = \{x \in X \mid d(x, x_0) < \varepsilon\}$. The ball $B_\varepsilon(x_0)$ is also called the ε-**neighborhood** of x_0.

A subset U of the metric space X is called **open** if for every point $x \in U$ there exists $\varepsilon > 0$ such that $B_\varepsilon(x) \subset U$.

Let (Y, ρ) be another metric space and $f : X \to Y$ be a function.

Proposition 1 The function f is continuous at a point $x \in X$ if and only if for every open set $W \subset Y$ containing $f(x)$ there is an open set $U \subset X$ containing x such that $f(U) \subset W$.

Proposition 2 The function f is continuous on the entire set X if and only if for any open set $W \subset Y$ the preimage $f^{-1}(W)$ is an open set in X.

Topological space

Definition. Given a nonempty set X, a **topology** on X is a collection $\mathcal{U}$ of subsets of X such that

- $\emptyset \in \mathcal{U}$ and $X \in \mathcal{U}$,
- any intersection of finitely many elements of $\mathcal{U}$ is also in $\mathcal{U}$,
- any union of elements of $\mathcal{U}$ is also in $\mathcal{U}$.

Elements of $\mathcal{U}$ are referred to as **open sets** of the topology. A set endowed with a topology is called a **topological space**.

We say that a sequence of points $x_1, x_2, \ldots$ of the topological space X **converges** to a point $y \in X$ if for every open set $U \in \mathcal{U}$ containing y there exists a natural number n_0 such that $x_n \in U$ for $n \geq n_0$.

Given another topological space Y and a function $f : X \to Y$, we say that f is **continuous** if for any open set $W \subset Y$ the preimage $f^{-1}(W)$ is an open set in X.

Examples of topological spaces

- *Metric space*

X: a metric space, $\mathcal{U}$: the set of all open subsets of X ($\mathcal{U}$ is referred to as the topology induced by the metric).

- *Trivial topology*

X: any nonempty set, $\mathcal{U} = \{\emptyset, X\}$.

- *Discrete topology*

X: any nonempty set, $\mathcal{U}$: the set of all subsets of X.

- *Subspace of a topological space*

X: nonempty subset of a topological space Y with a topology $\mathcal{W}$, $\mathcal{U} = \{U \cap X \mid U \in \mathcal{W}\}$.

Space of infinite sequences

Let $\mathcal{A}$ be a finite set. We denote by $\Sigma_{\mathcal{A}}$ the set of all infinite sequences $\mathbf{s} = (s_1 s_2 \dots)$, $s_i \in \mathcal{A}$. Elements of $\Sigma_{\mathcal{A}}$ are also referred to as **infinite words** over the **alphabet** $\mathcal{A}$.

For any finite sequence $s_1 s_2 \dots s_n$ of elements of $\mathcal{A}$ let $C(s_1 s_2 \dots s_n)$ denote the set of all infinite words $\mathbf{s} \in \Sigma_{\mathcal{A}}$ that begin with this sequence. The sets $C(s_1 s_2 \dots s_n)$ are called **cylinders**. Let $\mathcal{U}$ be the collection of all subsets of $\Sigma_{\mathcal{A}}$ that can be represented as unions of cylinders.

Proposition 1 $\mathcal{U}$ is a topology on $\Sigma_{\mathcal{A}}$.

The topological space $(\Sigma_{\mathcal{A}}, \mathcal{U})$ is **metrizable**, which means that the topology $\mathcal{U}$ is induced by a metric on $\Sigma_{\mathcal{A}}$. For any $\mathbf{s}, \mathbf{t} \in \Sigma_{\mathcal{A}}$ let $d(\mathbf{s}, \mathbf{t}) = 2^{-n}$ if $s_i = t_i$ for $1 \leq i \leq n$ while $s_{n+1} \neq t_{n+1}$. Also, let $d(\mathbf{s}, \mathbf{t}) = 0$ if $s_i = t_i$ for all $i \geq 1$.

Proposition 2 The function d is a metric on $\Sigma_{\mathcal{A}}$ that induces the topology $\mathcal{U}$.

Symbolic dynamics

The symbolic dynamics is concerned with the study of some continuous transformations of the topological space $\Sigma_{\mathcal{A}}$ of infinite words over a finite alphabet $\mathcal{A}$. The most important of them is the **shift** transformation $\sigma : \Sigma_{\mathcal{A}} \to \Sigma_{\mathcal{A}}$ defined by $\sigma(s_0 s_1 s_2 \dots) = (s_1 s_2 \dots)$.

Proposition The shift transformation is continuous.

Proof: We have to show that for any open set $W \subset \Sigma_{\mathcal{A}}$ the preimage $\sigma^{-1}(W)$ is also open. The set W is a union of cylinders: $W = \bigcup_{\beta \in B} C_\beta$. Since

$$\sigma^{-1}\Big(\bigcup_{\beta \in B} C_\beta\Big) = \bigcup_{\beta \in B} \sigma^{-1}(C_\beta),$$

it is enough to show that the preimage of any cylinder C_β is open. Let $C_\beta = C(s_1 s_2 \dots s_n)$. Then $\sigma^{-1}(C_\beta)$ is the union of cylinders $C(s_0 s_1 s_2 \dots s_n)$, $s_0 \in \mathcal{A}$, hence it is open.

Continuity of the itinerary map

Let $f : \mathbb{R} \to \mathbb{R}$ be a unimodal map, Λ be the set of all points $x \in \mathbb{R}$ such that $O_f^+(x) \subset [0, 1]$, and $S : \Lambda \to \Sigma_2 = \Sigma_{\{0,1\}}$ be the itinerary map.

Proposition The itinerary map S is continuous.

Proof: Since every open subset of Σ_2 is a union of cylinders, it is enough to show that for any cylinder $C = C(s_0 s_1 \dots s_n)$ the preimage $S^{-1}(C)$ is an open subset of Λ, i.e., $S^{-1}(C) = U \cap \Lambda$, where U is an open subset of $\mathbb{R}$.

Clearly, $S^{-1}(C) = I_{s_0 s_1 \dots s_n} \cap \Lambda$, where

$$I_{s_0 s_1 \dots s_n} = \{x \in [0, 1] \mid f^k(x) \in I_{s_k},\ 0 \le k \le n\}.$$

We know that $I_{s_0 s_1 \dots s_n}$ is a closed interval and the set Λ is covered by 2^{n+1} disjoint closed intervals of the form $I_{t_0 t_1 \dots t_n}$, where each t_i is 0 or 1. It follows that there exists an open interval U such that $S^{-1}(C) = I_{s_0 s_1 \dots s_n} \cap \Lambda = U \cap \Lambda$.