MATH 614

Dynamical Systems and Chaos

Lecture 7:
Compact sets.
Topological conjugacy (continued).
Definition of chaos (revisited).

Compact sets

Definition. A subset E of a topological space X is **compact** if any covering of E by open sets admits a finite subcover. The subset E is **sequentially compact** if any sequence of its elements has a subsequence converging to an element of E.

Proposition 1 For any set $E \subset X$, compactness implies sequential compactness. If the topological space X is metrizable, then the converse is true as well.

Proposition 2 Any closed subset of a compact set is also compact.

We say that a topological space X is **Hausdorff** if any two distinct elements of X have disjoint neighborhoods. It is easy to show that any metrizable topological space is Hausdorff.

Proposition 3 In a Hausdorff topological space, every compact set is closed.

Proposition 4 A subset of the Euclidean space $\mathbb{R}^n$ is compact if and only if it is closed and bounded.

Proposition 5 The topological space $\Sigma_{\mathcal{A}}$ of infinite words over a finite alphabet $\mathcal{A}$ is compact.

Proof: Since the topological space $\Sigma_{\mathcal{A}}$ is metrizable, it is enough to prove sequential compactness. Suppose $\mathbf{s}^{(1)}, \mathbf{s}^{(2)}, \dots$ is a sequence of infinite words over the alphabet $\mathcal{A}$. Note that a subsequence $\mathbf{s}^{(n_1)}, \mathbf{s}^{(n_2)}, \mathbf{s}^{(n_3)}, \dots$ converges to some $\mathbf{s} \in \Sigma_{\mathcal{A}}$ if and only if every finite beginning of $\mathbf{s}$ is also a beginning of $\mathbf{s}^{(n_k)}$ for k large enough.

Since $\mathcal{A}$ is a finite set, the number of finite words over $\mathcal{A}$ of any prescribed length is finite. It follows by induction that there exists a sequence of letters $s_1, s_2, \dots$ such that for any $k \in \mathbb{N}$ the finite word $s_1 s_2 \dots s_k$ occurs as a beginning of $\mathbf{s}^{(n)}$ for infinitely many n's. Then we choose indices $n_1 < n_2 < \dots$ so that $s_1 s_2 \dots s_k$ is a beginning of $\mathbf{s}^{(n_k)}$ for $k = 1, 2, \dots$ It follows that $\mathbf{s}^{(n_k)} \to \mathbf{s} = (s_1 s_2 s_3 \dots)$ as $k \to \infty$.

Compact sets and continuous maps

Proposition 6 The image of a compact set under a continuous map is also compact.

Proposition 7 Any continuous, real-valued function on a compact set attains its maximal and minimal values.

Proposition 8 Suppose that a continuous map $f : X \to Y$ is invertible. If the topological space X is compact and Y is Hausdorff, then the inverse map f^{-1} is continuous as well.

Proposition 9 Suppose (X, d) and (Y, ρ) are metric spaces. If X is compact then any continuous function $f : X \to Y$ is **uniformly continuous**, which means that for any $\varepsilon > 0$ there exists $\delta > 0$ such that $d(x, y) < \delta$ implies $\rho\big(f(x), f(y)\big) < \varepsilon$ for all $x, y \in X$.

Topological conjugacy

Suppose $f : X \to X$ and $g : Y \to Y$ are transformations of topological spaces.

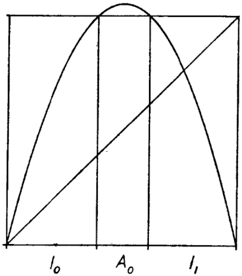
Definition. We say that a map $\phi : X \to Y$ is a **semi-conjugacy** of f with g if ϕ is onto and $\phi \circ f = g \circ \phi$.

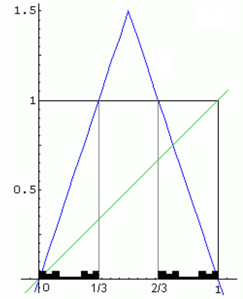
$$\begin{array}{ccc} X & \xrightarrow{f} & X \\ \phi\big\downarrow & & \big\downarrow\phi \\ Y & \xrightarrow{g} & Y \end{array}$$

The map ϕ is a **conjugacy** if, additionally, it is invertible. The map ϕ is a **topological conjugacy** if, additionally, it is a homeomorphism, which means that both ϕ and ϕ^{-1} are continuous. In the latter case, we say that the maps f and g are **topologically conjugate**. Note that $f = \phi^{-1} g \phi$ and $g = \phi f \phi^{-1}$.

Unimodal maps (revisited)

Let $f : \mathbb{R} \to \mathbb{R}$ be a unimodal map, Λ be the set of all points $x \in \mathbb{R}$ such that $O_f^+(x) \subset [0,1]$, and $S : \Lambda \to \Sigma_2 = \Sigma_{\{0,1\}}$ be the itinerary map.





Then S is a continuous semi-conjugacy of $f|_\Lambda$ with the shift. If S is one-to-one, then S is a topological conjugacy (since Λ and Σ_2 are compact sets).

Topological conjugacy of linear maps

Consider the family of linear maps $f_\lambda : \mathbb{R} \to \mathbb{R}$ given by $f_\lambda(x) = \lambda x,\ x \in \mathbb{R},$ where λ is a real parameter.

Let us also define another family of maps $\phi_\alpha : \mathbb{R} \to \mathbb{R}$ depending on a parameter $\alpha > 0$:

$$\phi_\alpha(x) = \begin{cases} x^\alpha & \text{if } x \geq 0, \\ -|x|^\alpha & \text{if } x < 0. \end{cases}$$

Note that ϕ_α is a homeomorphism and $\phi_\alpha^{-1} = \phi_{1/\alpha}$. For any $\lambda, x \geq 0$,

$$\phi_\alpha f_\lambda \phi_\alpha^{-1}(x) = \phi_\alpha f_\lambda(x^{1/\alpha}) = \phi_\alpha(\lambda x^{1/\alpha}) = (\lambda x^{1/\alpha})^\alpha = \lambda^\alpha x.$$

Since $f_\lambda(-x) = -f_\lambda(x)$ and $\phi_\alpha(-x) = -\phi_\alpha(x)$ for all x, the same equality holds for $\lambda \geq 0$ and $x < 0$. Similarly, for $\lambda < 0$ and any $x \in \mathbb{R}$ we obtain $\phi_\alpha f_\lambda \phi_\alpha^{-1}(x) = -|\lambda|^\alpha x$.

Therefore $\phi_\alpha f_\lambda \phi_\alpha^{-1} = f_{\lambda'}$, where $\lambda' = \phi_\alpha(\lambda)$.

Proposition Two linear maps f_λ and $f_{\lambda'}$ are topologically conjugate if and only if one of the following conditions holds: **(i)** $\lambda, \lambda' < -1$, **(ii)** $\lambda = \lambda' = -1$, **(iii)** $-1 < \lambda, \lambda' < 0$, **(iv)** $\lambda = \lambda' = 0$, **(v)** $0 < \lambda, \lambda' < 1$, **(vi)** $\lambda = \lambda' = 1$, **(vii)** $\lambda, \lambda' > 1$.

Proof: If one of the seven conditions holds, then $\lambda' = \phi_\alpha(\lambda)$ for some $\alpha > 0$. It follows that $\phi_\alpha f_\lambda \phi_\alpha^{-1} = f_{\lambda'}$, in particular, f_λ and $f_{\lambda'}$ are topologically conjugate.

If neither condition holds, we need to distinguish f_λ from $f_{\lambda'}$ by a property invariant under topological conjugacy. First notice that f_0 is the only linear map that is not one-to-one. Further, f_1 is the identity map and f_{-1} is distinguished since f_{-1}^2 is the identity map while f_{-1} is not. The only fixed point 0 of f_λ is attracting if $|\lambda| < 1$ and repelling if $|\lambda| > 1$. Finally, for any $x \neq 0$ the interval with endpoints x and $f_\lambda(x)$ contains the fixed point 0 if $\lambda < 0$ and does not if $\lambda > 0$.

Proposition 1 Suppose $f : [0, a] \to \mathbb{R}$ and $g : [0, b] \to \mathbb{R}$ are continouos maps such that $f(0) = g(0) = 0$, $f(x) < x$ for $0 < x \leq a$, and $g(x) < x$ for $0 < x \leq b$. Then f and g are topologically conjugate.

Let $U = (f(a), a)$. Then U is a **wandering domain** of the map f, which means that sets $U, f(U), f^2(U), \dots$ are disjoint. Similarly, $V = (g(b), b)$ is a wandering domain of g.

$$\begin{array}{ccccccc} U & \xrightarrow{f} & f(U) & \xrightarrow{f} & f^2(U) & \xrightarrow{f} & \dots \\ \phi \big\downarrow & & & & & & \\ V & \xrightarrow{g} & g(V) & \xrightarrow{g} & g^2(V) & \xrightarrow{g} & \dots \end{array}$$

Proposition 2 Suppose $f, g : \mathbb{R} \to \mathbb{R}$ are continuously differentiable maps such that $f(0) = g(0) = 0$, $0 < f'(x) < 1$ and $0 < g'(x) < 1$ for all $x \in \mathbb{R}$. Then f and g are topologically conjugate.

Topological transitivity

Suppose $f : X \to X$ is a continuous transformation of a topological space X.

Definition. The map f is **topologically transitive** if for any nonempty open sets $U, V \subset X$ there exists a natural number n such that $f^n(U) \cap V \neq \emptyset$.

$$U \ni x \longmapsto f(x) \longmapsto f^2(x) \longmapsto \cdots \longmapsto f^k(x) \in V$$

Topological transitivity means that the dynamical system f is, in a sense, indecomposable.

Proposition 1 Topological transitivity is preserved under topological conjugacy.

Proposition 2 If the map f has a dense orbit, then it is topologically transitive.

Proposition 3 If X is a metrizable compact space, then any topologically transitive transformation of X has a dense orbit.

Separation of orbits

Suppose $f : X \to X$ is a continuous transformation of a metric space (X, d).

Definition. We say that f has **sensitive dependence on initial conditions** if there is $\delta > 0$ such that, for any $x \in X$ and a neighborhood U of x, there exist $y \in U$ and $n \geq 0$ satisfying $d\big(f^n(y), f^n(x)\big) > \delta$.

We say that the map f is **expansive** if there is $\delta > 0$ such that, for any $x, y \in X$, $x \neq y$, there exists $n \geq 0$ satisfying $d\big(f^n(y), f^n(x)\big) > \delta$.

Proposition If X is compact, then changing the metric d to another metric that induces the same topology cannot affect sensitive dependence on i.c. and expansiveness of the map f.

Corollary For continuous transformations of compact metric spaces, sensitive dependence on initial conditions and expansiveness are preserved under topological conjugacy.

Definition of chaos

Suppose $f : X \to X$ is a continuous transformation of a metric space (X, d).

Definition. We say that the map f is **chaotic** if

- f has sensitive dependence on initial conditions;
- f is topologically transitive;
- periodic points of f are dense in X.

Proposition 1 For continuous transformations of compact metric spaces, chaoticity is preserved under topological conjugacy.

Proposition 2 The shift $\sigma : \Sigma_{\mathcal{A}} \to \Sigma_{\mathcal{A}}$ is chaotic.

Corollary Any dynamical system topologically conjugate to the shift is chaotic.