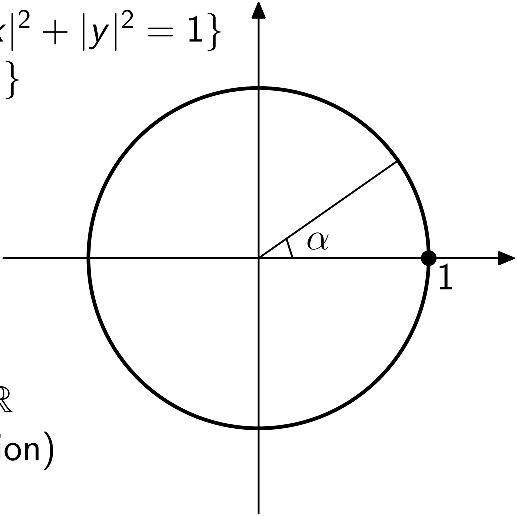
MATH 614

Dynamical Systems and Chaos

Lecture 11: Maps of the circle.

Circle S^1.

$S^1 = \{(x,y) \in \mathbb{R}^2 : |x|^2 + |y|^2 = 1\}$
$S^1 = \{z \in \mathbb{C} : |z| = 1\}$
$\mathbb{T}^1 = \mathbb{R}/\mathbb{Z}$
$\mathbb{T}^1 = \mathbb{R}/2\pi\mathbb{Z}$



$\alpha : S^1 \to [0, 2\pi)$,
angular coordinate

$\alpha : S^1 \to \mathbb{R}/2\pi\mathbb{Z} \to \mathbb{R}$
(multi-valued function)

$\phi : \mathbb{R} \to S^1$,
$\phi(x) = (\cos x, \sin x), \quad S^1 \subset \mathbb{R}^2$.
$\phi(x) = e^{ix} = \cos x + i \sin x, \quad S^1 \subset \mathbb{C}$.
ϕ: wrapping map
$\phi(x + 2\pi k) = \phi(x), \; k \in \mathbb{Z}$.

$\alpha \in \mathbb{R}$ is an angular coordinate of $x \in S^1$ if and only if $\phi(\alpha) = x$.

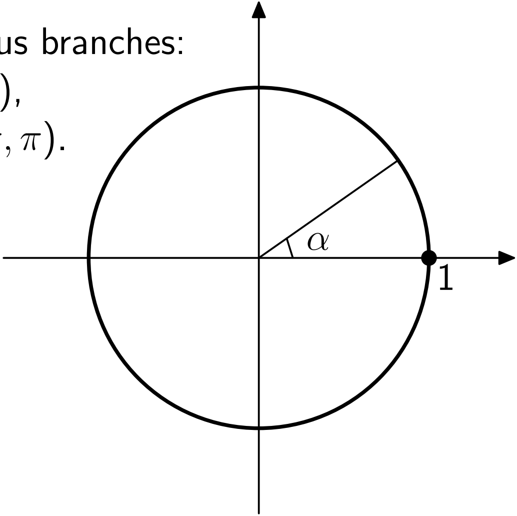
For any arc $\gamma \subset S^1$ there exists a continuous branch $\alpha : \gamma \to \mathbb{R}$ of the angular coordinate.

If $\alpha_1 : \gamma \to \mathbb{R}$ and $\alpha_2 : \gamma \to \mathbb{R}$ are two continuous branches then $\alpha_1 - \alpha_2$ is a constant $2\pi k$, $k \in \mathbb{Z}$.

Examples of continuous branches:

$\alpha : S^1 \setminus \{1\} \to (0, 2\pi)$,

$\alpha : S^1 \setminus \{-1\} \to (-\pi, \pi)$.



$f : S^1 \to S^1$, continuous map

Example. $D : z \mapsto z^2$ (**doubling map**)
in angular coordinates: $\alpha \mapsto 2\alpha \pmod{2\pi}$.

The doubling map: smooth, 2-to-1, no critical points.

Theorem The doubling map is chaotic.

Orientation-preserving and orientation-reversing

The real line $\mathbb{R}$ has two orientations.

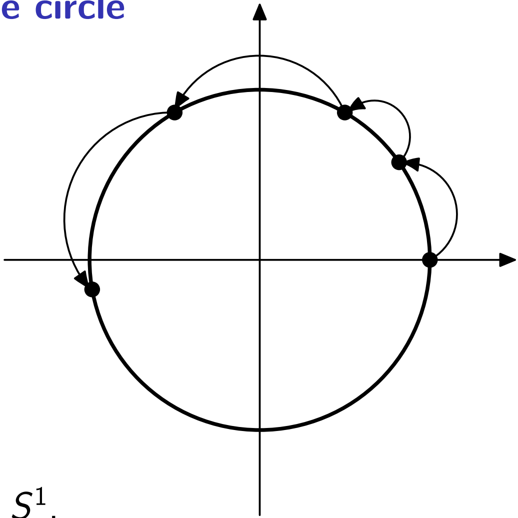
For maps of an interval:
orientation-preserving = monotone increasing,
orientation-reversing = monotone decreasing.

The circle S^1 also has two orientations (clockwise and counterclockwise).

Given a map $f : S^1 \to S^1$, we say that a map $F : \mathbb{R} \to \mathbb{R}$ is a **lift** of f if $f \circ \phi = \phi \circ F$, where $\phi : \mathbb{R} \to S^1$ is the wrapping map. Any continuous map $f : S^1 \to S^1$ admits a continuous lift F. The lift satisfies $F(x + 2\pi) - F(x) = 2\pi k$ for some $k \in \mathbb{Z}$ and all $x \in \mathbb{R}$. If F_0 is another continuous lift of f, then $F - F_0$ is a constant function.

A continuous map $f : S^1 \to S^1$ is **orientation-preserving** (resp., **orientation-reversing**) if so is the continuous lift of f.

Maps of the circle



$f : S^1 \to S^1$,

f an orientation-preserving homeomorphism.

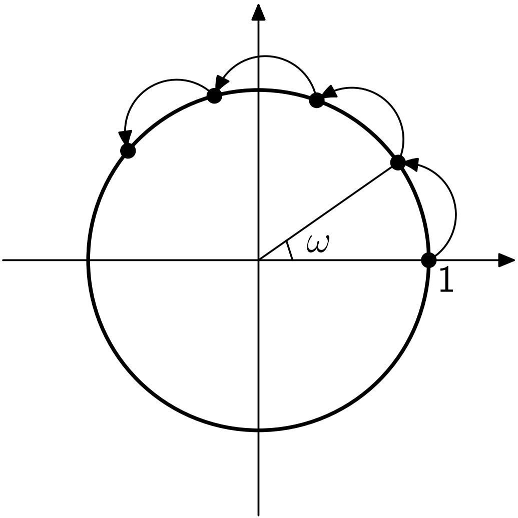
Rotations of the circle

$R_\omega : S^1 \to S^1$, rotation by angle $\omega \in \mathbb{R}$.
$R_\omega(z) = e^{i\omega} z$, complex coordinate z;
$R_\omega(\alpha) = \alpha + \omega \pmod{2\pi}$, angular coordinate α.

Each R_ω is an orientation-preserving diffeomorphism;
each R_ω is an isometry;
each R_ω preserves Lebesgue measure on S^1.

R_ω is a one-parameter family of maps.
R_ω is a **transformation group**.

Indeed, $R_{\omega_1} R_{\omega_2} = R_{\omega_1+\omega_2}$, $R_\omega^{-1} = R_{-\omega}$.
It follows that $R_\omega^n = R_{n\omega}$, $n = 1, 2, \dots$.
Also, $R_0 = \mathrm{id}$ and $R_{\omega+2\pi k} = R_\omega$, $k \in \mathbb{Z}$.


ω
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An angle ω is called **rational** if $\omega = r\pi,\ r \in \mathbb{Q}$. Otherwise ω is an **irrational** angle.

If ω is a rational angle then R_ω is a periodic map. All points of S^1 are periodic of the same period.

If $\omega = 2\pi m/n,$ where m and n are coprime integers, $n > 0,$ then the period of R_ω is n.

If ω is irrational then R_ω has no periodic points.

If ω is irrational then R_ω is **minimal**: each orbit is dense in S^1.

If ω is irrational then each orbit of R_ω is **uniformly distributed** in S^1.

Minimality

Theorem Suppose ω is an irrational angle. Then the rotation R_ω is minimal: all orbits of R_ω are dense in S^1.

Proof: Take an arc $\gamma \subset S^1$. Then $R_\omega^n(\gamma)$, $n \geq 1$, is an arc of the same length as γ. Since S^1 has finite length, the arcs $\gamma, R_\omega(\gamma), R_\omega^2(\gamma), \dots$ cannot all be disjoint. Hence $R_\omega^n(\gamma) \cap R_\omega^m(\gamma) \neq \emptyset$ for some $0 \leq n < m$. But $R_\omega^n(\gamma) \cap R_\omega^m(\gamma) = R_\omega^n(\gamma \cap R_\omega^{m-n}(\gamma))$ so $\gamma \cap R_\omega^{m-n}(\gamma) \neq \emptyset$.

Thus for any $\varepsilon > 0$ there exists $k \geq 1$ such that $R_\omega^k = R_{k\omega}$ is the rotation by an angle ω', $|\omega'| < \varepsilon$. Note that $\omega' \neq 0$ since ω is an irrational angle. Pick any $x \in S^1$. Let $n = \lceil 2\pi/|\omega'| \rceil$. Then points $x, R_{k\omega}(x), R_{k\omega}^2(x), \dots, R_{k\omega}^n(x)$ divide S^1 into arcs of length $< \varepsilon$.

Uniform distribution

Let $T: S^1 \to S^1$ be a homeomorphism and $x \in S^1$. Consider the orbit $x, T(x), T^2(x), \dots, T^n(x), \dots$

Let $\gamma \subset S^1$ be an arc. By $N(x, \gamma; n)$ denote the number of integers $k \in \{0, 1, \dots, n-1\}$ such that $T^k(x) \in \gamma$. The orbit of x is **uniformly distributed** in S^1 if

$$\lim_{n\to\infty} \frac{N(x, \gamma_1; n)}{N(x, \gamma_2; n)} = 1$$

for any two arcs γ_1 and γ_2 of the same length.

An equivalent condition:

$$\lim_{n\to\infty} \frac{N(x,\gamma_1;n)}{N(x,\gamma_2;n)} = \frac{length(\gamma_1)}{length(\gamma_2)}$$

for any arcs γ_1 and γ_2.

Another equivalent condition:

$$\lim_{n\to\infty} \frac{N(x,\gamma;n)}{n} = \frac{length(\gamma)}{2\pi}$$

for any arc γ.

Theorem Suppose ω is an irrational angle. Then all orbits of the rotation R_ω are uniformly distributed in S^1.

Fractional linear transformations of S^1

A **fractional linear transformation** of the complex plane $\mathbb{C}$ is given by

$$f(z) = \frac{az+b}{cz+d}, \qquad a, b, c, d \in \mathbb{C}.$$

How can we tell if $f(S^1) = S^1$? This happens in the case

$$f(z) = e^{i\psi} \frac{z - z_0}{\bar{z}_0 z - 1},$$

where $|z_0| \neq 1$ and $\psi \in \mathbb{R}$. Indeed, if $z \in S^1$ then

$z = e^{i\alpha}$, $z_0 = re^{i\beta}$,

$z - z_0 = e^{i\alpha} - re^{i\beta} = e^{i\alpha}(1 - re^{i\beta}e^{-i\alpha})$,

$\bar{z}_0 z - 1 = re^{-i\beta}e^{i\alpha} - 1$ so that $f(z) \in S^1$.

Fractional linear transformations of S^1

$S^1 = \{z \in \mathbb{C} : |z| = 1\}$,
$f : S^1 \to S^1$,

$$f(z) = -e^{i\omega} \frac{z - z_0}{\bar{z}_0 z - 1},$$

where $z \in \mathbb{C}$, $|z_0| \neq 1$ and $\omega \in \mathbb{R}$.

Fractional linear transformations of S^1 form a **group**. Rotations of the circle form a **subgroup** ($z_0 = 0$).

f is orientation-preserving if $|z_0| < 1$ and orientation-reversing if $|z_0| > 1$.

$$f(z) = \frac{az+b}{cz+d}, \quad g(z) = \frac{a'z+b'}{c'z+d'},$$

$$f(g(z)) = \frac{a\frac{a'z+b'}{c'z+d'}+b}{c\frac{a'z+b'}{c'z+d'}+d} = \frac{(aa'+bc')z+ab'+bd'}{(ca'+dc')z+cb'+dd'},$$

$$\frac{az+b}{cz+d} \mapsto \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

Composition of fractional linear transformations corresponds to matrix multiplication.

$$f(z) = -e^{i\omega}\frac{z - z_0}{\bar{z}_0 z - 1},$$

$$-e^{i\omega/2}\begin{pmatrix} e^{i\omega/2} & -z_0 e^{i\omega/2} \\ -\bar{z}_0 e^{-i\omega/2} & e^{-i\omega/2} \end{pmatrix}.$$

$\det = 1 - |z_0|^2, \;\; \mathrm{Tr} = e^{i\omega/2} + e^{-i\omega/2} = 2\cos(\omega/2).$

Characteristic equation:
$\lambda^2 - 2\cos(\omega/2)\lambda + 1 - |z_0|^2 = 0.$

Discriminant:
$D = \cos^2(\omega/2) - 1 + |z_0|^2 = |z_0|^2 - \sin^2(\omega/2).$

If $D < 0$ then f is **elliptic**.
If $D = 0$ then f is **parabolic**.
If $D > 0$ then f is **hyperbolic**.

Theorem (i) If f is elliptic then f has no fixed points and is topologically conjugate to a rotation.
(ii) If f is parabolic then f has a unique fixed point, which is neutral. Besides, the fixed point is weakly semi-attracting and semi-repelling.
(iii) If f is hyperbolic then f has two fixed points; one is attracting, the other is repelling.

Example. Given $\omega \in (0, \pi)$, the one-parameter family

$$f_r(z) = e^{i\omega} \frac{z - r}{1 - rz}, \quad 0 \leq r < 1$$

undergoes a saddle-node bifurcation at $r = r_0 = |\sin(\omega/2)|$.

Rotation number

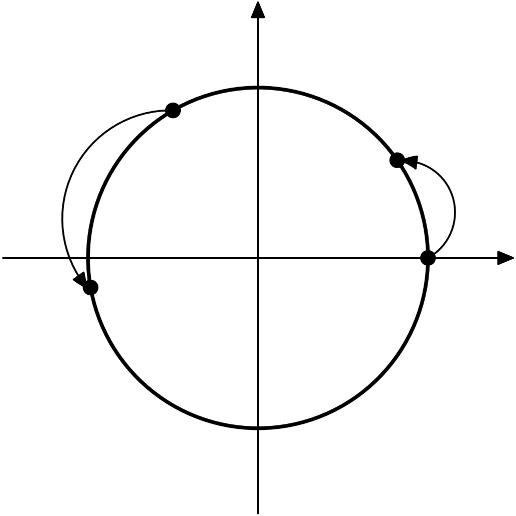
Suppose $T : S^1 \to S^1$ is an orientation-preserving homeomorphism.

Is T topologically conjugate to a rotation R_ω?

Assume this is so, then how can we find ω?

For any $x \in S^1$ let $\omega(T, x)$ denote the length of the shortest arc that goes from x to $T(x)$ in the counterclockwise direction.

If T is a rotation then $\omega(T, x)$ is a constant.



Consider the average

$$A_n(T,x) = \frac{1}{n}\sum_{k=0}^{n-1}\omega(T, T^k(x)).$$

Theorem The limit

$$\lim_{n\to\infty} A_n(T,x)$$

exists for any $x \in S^1$ and does not depend on x.

The **rotation number** of T is

$$\rho(T) = \frac{1}{2\pi}\lim_{n\to\infty} A_n(T,x).$$

For any T, $0 \leq \rho(T) < 1$.
$\rho(R_\omega) = \omega/(2\pi) \pmod 1$.

Proposition 1 If T_1 and T_2 are topologically conjugate then $\rho(T_1) = \pm\rho(T_2) \pmod 1$.

Proposition 2 $\rho(T^n) = n\rho(T) \pmod 1$.

Proposition 3 If T has a fixed point then $\rho(T) = 0$.

Proposition 4 If T has a periodic point of period n then $\rho(T) = k/n$, where $k \in \mathbb{Z}$, $0 \leq k < n$.

Proposition 5 If $\rho(T) = 0$ then T has a fixed point.

Proof: Suppose T has no fixed points. Then $0 < \omega(T, x) < 2\pi$ for any $x \in S^1$. Since $\omega(T, x)$ is a continuous function of x, there exists $\varepsilon > 0$ such that $\varepsilon \leq \omega(T, x) \leq 2\pi - \varepsilon$ for any $x \in S^1$. Then $\varepsilon \leq A_n(T, x) \leq 2\pi - \varepsilon$ for all $x \in S^1$ and $n = 1, 2, \ldots$. It follows that $\varepsilon/(2\pi) \leq \rho(T) \leq 1 - \varepsilon/(2\pi)$.

Proposition 6 If $\rho(T)$ is rational then T has a periodic point.

Theorem (Denjoy) If T is C^2 smooth and the rotation number $\rho(T)$ is irrational, then T is topologically conjugate to a rotation of the circle.

Example (Denjoy). There exists C^1 smooth diffeomorphism T of S^1 such that $\rho(T)$ is irrational but T is not minimal.