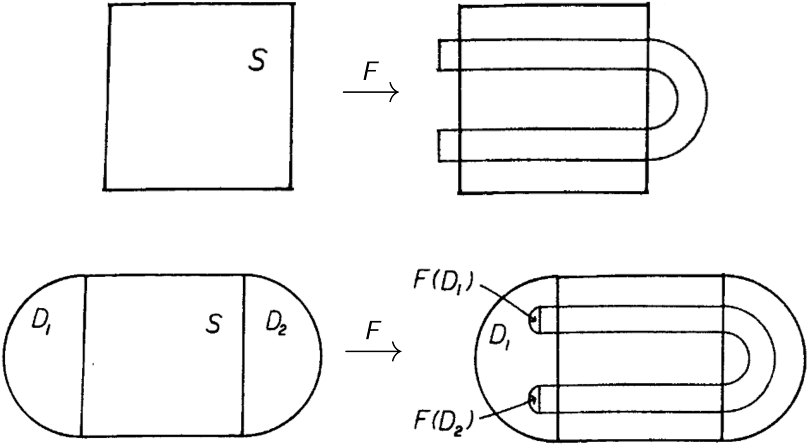
MATH 614

Dynamical Systems and Chaos

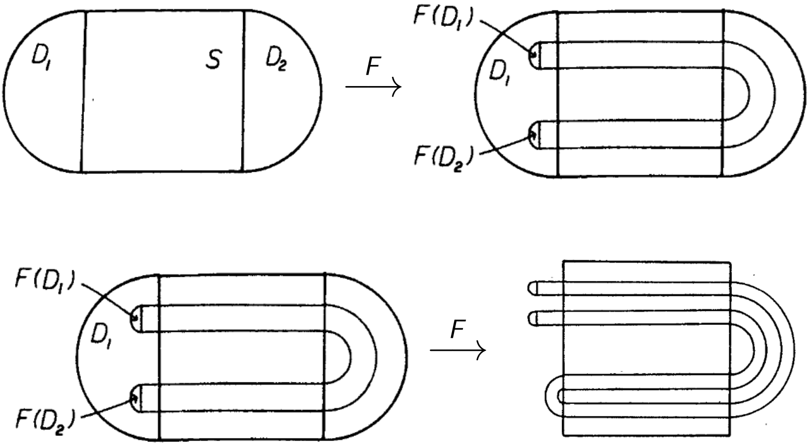
Lecture 14:
The horseshoe map.
Invertible symbolic dynamics.
Stable and unstable sets.

The Smale horseshoe map

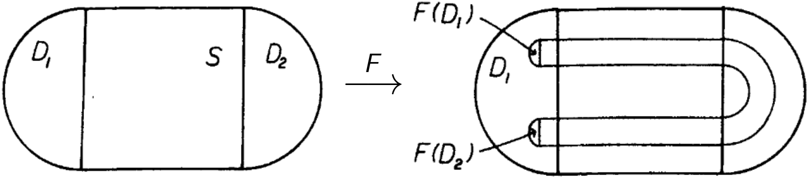
Stephen Smale, 1960



The Smale horseshoe map

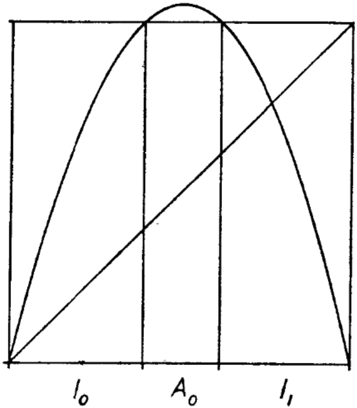


The Smale horseshoe map



The map F is contracting on D_1 and $F(D_1) \subset D_1$. It follows that there is a unique fixed point $p \in D_1$ and the orbit of any point in D_1 converges to p.

Moreover, any orbit that leaves the square S converges to p.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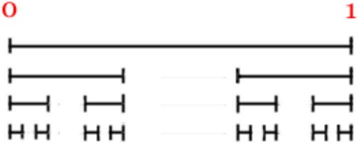
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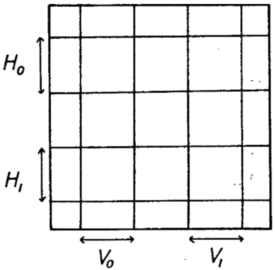
Itineraries



$$F^{-1}(S) = V_0 \cup V_1, \;\; F(V_0) = H_0, \;\; F(V_1) = H_1.$$

Let Λ_1 be the set of all points in S whose orbits stay in S. Let Λ_2 be the set of all points in S with infinite backward orbit. Finally, let $\Lambda = \Lambda_1 \cap \Lambda_2$.

For any point in Λ_1 we can define the forward itinerary while for any point in Λ_2 we can define the backward itinerary. For any $x \in \Lambda$ we can define the full itinerary $S(x) = (\ldots s_{-2}s_{-1}.s_0s_1s_2\ldots)$. Then $S(F(x)) = (\ldots s_{-2}s_{-1}s_0.s_1s_2\ldots)$.

Bi-infinite words

Given a finite set $\mathcal{A}$ (an alphabet), we denote by $\Sigma_{\mathcal{A}}^{\pm}$ the set of all **bi-infinite words** over $\mathcal{A}$, i.e., bi-infinite sequences $\mathbf{s} = (\dots s_{-2}s_{-1}.s_0s_1s_2\dots)$, $s_i \in \mathcal{A}$. Any bi-infinite word in $\Sigma_{\mathcal{A}}^{\pm}$ comes with the standard numbering of letters determined by the decimal point.

For any finite words w_-, w_+ over the alphabet $\mathcal{A}$, we define a **cylinder** $C(w_-, w_+)$ to be the set of all bi-infinite words $\mathbf{s} \in \Sigma_{\mathcal{A}}^{\pm}$ of the form $(\dots s_{-2}s_{-1}w_-.w_+s_1s_2\dots)$, $s_i \in \mathcal{A}$. The topology on $\Sigma_{\mathcal{A}}^{\pm}$ is defined so that open sets are unions of cylinders. Two bi-infinite words are considered close in this topology if they have a long common part around the decimal point.

The topological space $\Sigma_{\mathcal{A}}^{\pm}$ is metrizable. A compatible metric is defined as follows. For any $\mathbf{s}, \mathbf{t} \in \Sigma_{\mathcal{A}}^{\pm}$ we let $d(\mathbf{s}, \mathbf{t}) = 2^{-n}$ if $s_i = t_i$ for $0 \le |i| < n$ while $s_n \neq t_n$ or $s_{-n} \neq t_{-n}$. Also, let $d(\mathbf{s}, \mathbf{t}) = 0$ if $\mathbf{s} = \mathbf{t}$.

Invertible symbolic dynamics

The **shift** transformation $\sigma : \Sigma_{\mathcal{A}}^{\pm} \to \Sigma_{\mathcal{A}}^{\pm}$ is defined by $\sigma(\dots s_{-2}s_{-1}.s_0s_1s_2\dots) = (\dots s_{-2}s_{-1}s_0.s_1s_2\dots)$. It is also called the **two-sided shift** while the shift on $\Sigma_{\mathcal{A}}$ is called the **one-sided shift**.

Proposition 1 The two-sided shift is a homeomorphism.

Proposition 2 Periodic points of σ are dense in $\Sigma_{\mathcal{A}}^{\pm}$.

Proposition 3 The two-sided shift admits a dense orbit.

Proposition 4 The two-sided shift is chaotic.

Proposition 5 The itinerary map $S : \Lambda \to \Sigma_{\mathcal{A}}^{\pm}$ of the horseshoe map is a homeomorphism.

Proposition 6 The topological spaces $\Sigma_{\mathcal{A}}^{\pm}$ and $\Sigma_{\mathcal{A}}$ are homeomorphic.

Stable and unstable sets

Let $f : X \to X$ be a continuous map of a metric space (X, d).

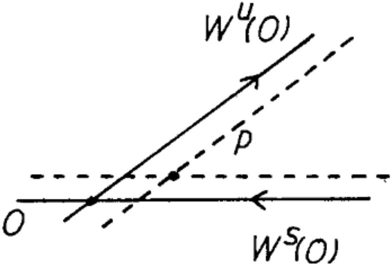
Definition. Two points $x, y \in X$ are **forward asymptotic** with respect to f if $d\big(f^n(x), f^n(y)\big) \to 0$ as $n \to \infty$. The **stable set** of a point $x \in X$, denoted $W^s(x)$, is the set of all points forward asymptotic to x.

Being forward asymptotic is an equivalence relation on X. The stable sets are equivalence classes of this relation. In particular, these sets form a partition of X.

In the case f is a homeomorphism, we say that two points $x, y \in X$ are **backward asymptotic** with respect to f if $d\big(f^{-n}(x), f^{-n}(y)\big) \to 0$ as $n \to \infty$. The **unstable set** of a point $x \in X$, denoted $W^u(x)$, is the set of all points backward asymptotic to x. The unstable set $W^u(x)$ coincides with the stable set of x relative to the inverse map f^{-1}.

Examples

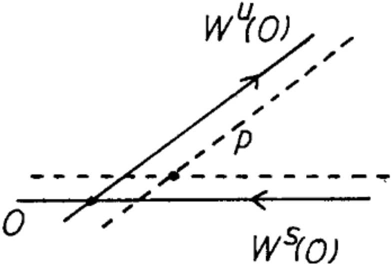
- Linear map $L : \mathbb{R}^n \to \mathbb{R}^n$.



The stable and unstable sets of the origin, $W^s(\mathbf{0})$ and $W^u(\mathbf{0})$, are transversal subspaces of the vector space $\mathbb{R}^n$. For any point $\mathbf{p} \in \mathbb{R}^n$, the stable and unstable sets are obtained from $W^s(\mathbf{0})$ and $W^u(\mathbf{0})$ by a translation: $W^s(\mathbf{p}) = \mathbf{p} + W^s(\mathbf{0})$, $W^u(\mathbf{p}) = \mathbf{p} + W^u(\mathbf{0})$.

Examples

- Hyperbolic toral automorphism $L_A : \mathbb{T}^2 \to \mathbb{T}^2$.



Stable and unstable sets of L_A are images of the corresponding sets of the linear map $L(\mathbf{x}) = A\mathbf{x}$, $\mathbf{x} \in \mathbb{R}^2$, under the natural projection $\pi : \mathbb{R}^2 \to \mathbb{T}^2$. These sets are dense in the torus $\mathbb{T}^2$.

Examples

- One-sided shift $\sigma : \Sigma_{\mathcal{A}} \to \Sigma_{\mathcal{A}}$.

Two infinite words $\mathbf{s} = (s_1 s_2 \dots)$ and $\mathbf{t} = (t_1 t_2 \dots)$ are forward asymptotic if they eventually coincide: $s_n = t_n$ for large n, in which case the orbits of $\mathbf{s}$ and $\mathbf{t}$ under the shift eventually coincide as well. The stable set $W^s(\mathbf{s})$ consists of all infinite words that differ from $\mathbf{s}$ in only finitely many letters. The set $W^s(\mathbf{s})$ is countable and dense in $\Sigma_{\mathcal{A}}$.

- Two-sided shift $\sigma : \Sigma_{\mathcal{A}}^{\pm} \to \Sigma_{\mathcal{A}}^{\pm}$.

Two bi-infinite words $\mathbf{s} = (\dots s_{-2} s_{-1} . s_0 s_1 s_2 \dots)$ and $\mathbf{t} = (\dots t_{-2} t_{-1} . t_0 t_1 t_2 \dots)$ are forward asymptotic if there exists $n_0 \in \mathbb{Z}$ such that $s_n = t_n$ for $n \geq n_0$. They are backward asymptotic if there exists $n_0 \in \mathbb{Z}$ such that $s_n = t_n$ for $n \leq n_0$.

Examples

- The horseshoe map $F : D \to D$.