MATH 614

Dynamical Systems and Chaos

Lecture 15: Markov partitions. Solenoid.

General symbolic dynamics

Suppose $f : X \to X$ is a dynamical system. Given a partition of the set X into disjoint subsets X_α, $\alpha \in \mathcal{A}$ indexed by elements of a finite set $\mathcal{A}$, we can define the (forward) **itinerary map** $S : X \to \Sigma_{\mathcal{A}}$ so that $S(x) = (s_0 s_1 s_2 \dots)$, where $f^n(x) \in X_{s_n}$ for all $n \geq 0$.

If the map f is invertible, then we can define the full itinerary map $S : X \to \Sigma^{\pm}_{\mathcal{A}}$.

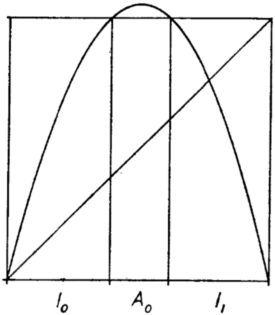
In the case f is continuous, the itinerary map is continuous if the sets X_α are **clopen** (i.e., both closed and open). If, additionally, X is compact, then the itinerary map provides a semi-conjugacy of f with a subshift.

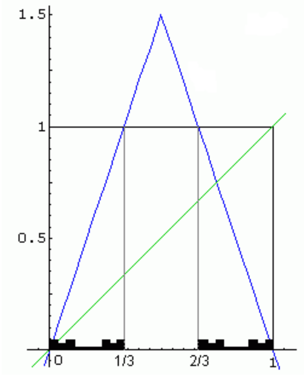
General symbolic dynamics

A more general construction is to take disjoint sets $X_\alpha,\ \alpha \in \mathcal{A}$ that need not cover the entire set X. Then the itinerary map is defined on a subset of X consisting of all points whose orbits stay in the union of the sets X_α.

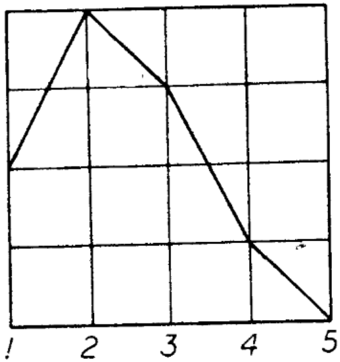
In the case X is an interval, a partition into clopen sets is not possible. Instead, we choose the sets X_α to be closed intervals with disjoint interiors. Then the itinerary map is not (uniquely) defined on a countable set.

Examples



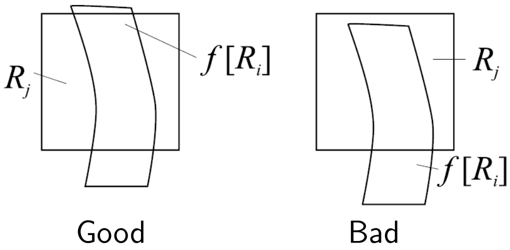


Examples



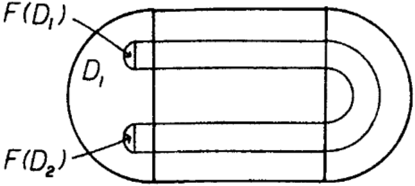
Markov partitions

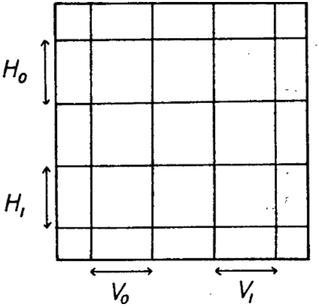
Definition. Given a metric space M and a continuous map $f : M \to M$, a **Markov partition** of M is partition of M into "rectangles" $\{R_1, \ldots, R_m\}$ such that whenever $x \in R_i$ and $f(x) \in R_j$, we have $f\big(W^u(x) \cap R_i\big) \supset W^u(f(x)) \cap R_j$ and $f\big(W^s(x) \cap R_i\big) \subset W^s(f(x)) \cap R_j$.



The condition ensures that all points in $W^s(x) \cap R_i$ have the same forward itinerary while all points in $W^u(x) \cap R_i$ have the same backward itinerary.

Examples

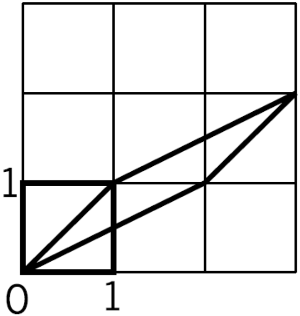


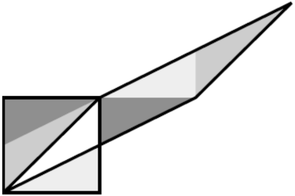


Cat map

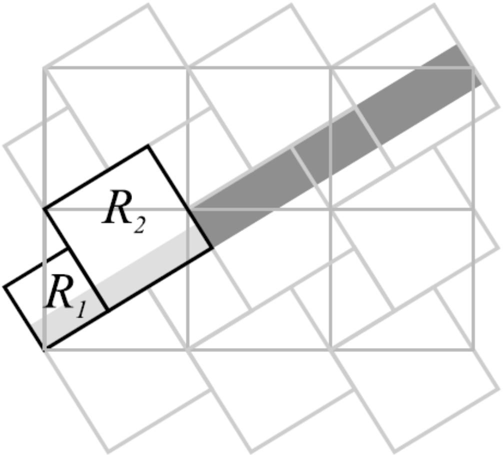
The **cat map** is a hyperbolic toral automorphism

$L_A : \mathbb{T}^2 \to \mathbb{T}^2$ given by the matrix $A = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$.





Markov partition for the cat map



Translations of the torus

For any vector $\mathbf{v} \in \mathbb{R}^n$ and a point of the n-dimensional torus $\mathbf{x} \in \mathbb{T}^n$, the sum $\mathbf{x} + \mathbf{v}$ is a well-defined element of $\mathbb{T}^n$.

Given $\mathbf{v} \in \mathbb{R}^n$, let $T_{\mathbf{v}}(\mathbf{x}) = \mathbf{x} + \mathbf{v}$ be the **translation** of the torus $\mathbb{T}^n$.

Theorem 1 Let $\mathbf{v} = (v_1, v_2, \dots, v_n)$. The linear flow $T_{t\mathbf{v}}$, $t \in \mathbb{R}$ is minimal (all orbits are dense) if and only if the real numbers $v_1, v_2, \dots, v_n$ are linearly independent over $\mathbb{Q}$. That is, if $r_1 v_1 + \cdots + r_n v_n = 0$ implies $r_1 = \cdots = r_n = 0$ for all $r_1, \dots, r_n \in \mathbb{Q}$.

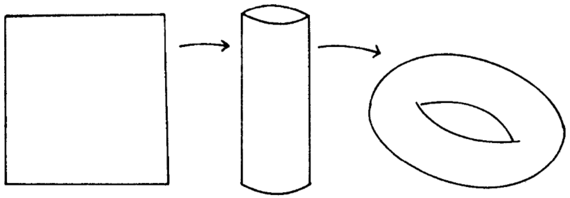
Theorem 2 Let $\mathbf{v} = (v_1, v_2, \dots, v_n)$. The translation $T_{\mathbf{v}}$ is minimal (all orbits are dense) if and only if the real numbers $1, v_1, v_2, \dots, v_n$ are linearly independent over $\mathbb{Q}$.

Solid torus

Let S^1 be the circle and B^2 be the unit disk in $\mathbb{R}^2$:

$$B^2 = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 \leq 1\}.$$

The Cartesian product $D = S^1 \times B^2$ is called the **solid torus**. It is a 3-dimensional manifold with boundary that can be realized as a closed subset in $\mathbb{R}^3$. The boundary ∂D is the torus.



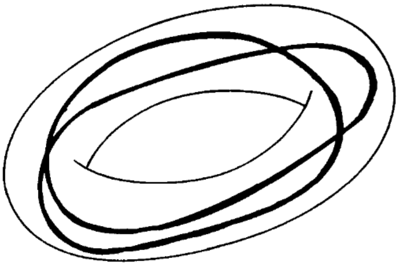
Let $D = S^1 \times B^2$ be the solid torus. We represent the circle S^1 as $\mathbb{R}/\mathbb{Z}$. For any $\theta \in S^1$ and $p \in B^2$ let

$$F(\theta, p) = \big(2\theta, ap + b\phi(\theta)\big),$$

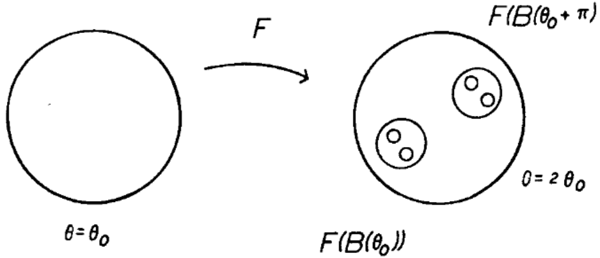
where $\phi : S^1 \to \partial B^2$ is defined by

$$\phi(\theta) = \big(\cos(2\pi\theta), \sin(2\pi\theta)\big)$$

and constants a, b are chosen so that $0 < a < b$ and $a + b < 1$. Then $F : D \to D$ is a smooth, one-to-one map. The image $F(D)$ is contained strictly inside of D.



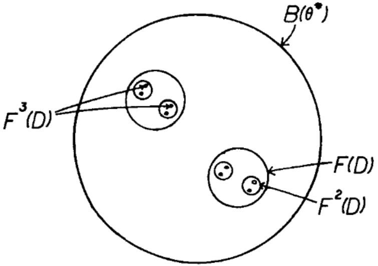
The solid torus $D = S^1 \times B^2$ is foliated by discs $B(\theta) = \{\theta\} \times B^2$. The image $F(B(\theta))$ is a smaller disc inside of $B(2\theta)$.



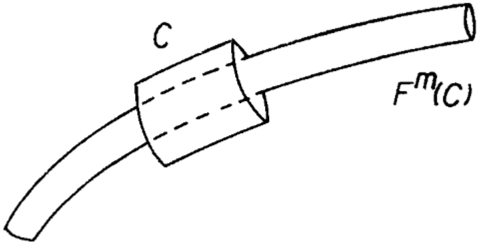
It follows that all points in a disc $B(\theta)$ are forward asymptotic. In particular, $B(\theta)$ is contained in the stable set $W^s(\mathbf{x})$ of any point $\mathbf{x} \in B(\theta)$. In fact, $W^s(\mathbf{x}) = \bigcup_{n,k\in\mathbb{Z}} B(\theta + n/2^k)$.

Solenoid

The sets $D, F(D), F^2(D), \ldots$ are closed and nested. The intersection $\Lambda = \bigcap_{n \geq 0} F^n(D)$ is called the **solenoid**.



The solenoid Λ is a compact set invariant under the map F. The restriction of F to Λ is an invertible map. The intersection of Λ with any disc $B(\theta)$ is a Cantor set. Moreover, Λ is locally the Cartesian product of a Cantor set and an arc.


C
$F^m(C)$

Properties of the solenoid

Theorem 1 The restriction $F|_\Lambda$ is chaotic, i.e.,

- it has sensitive dependence on initial conditions,
- periodic points are dense in Λ,
- it is topologically transitive.

Theorem 2 The solenoid Λ is an attractor of the map F. In particular, $\mathrm{dist}\big(F^n(\mathbf{x}), \Lambda\big) \to 0$ as $n \to \infty$ for all $\mathbf{x} \in D$.

Theorem 3 For any point $\mathbf{x} \in \Lambda$, the unstable set $W^u(\mathbf{x})$ is a smooth curve that is dense in Λ.

Theorem 4 The solenoid is connected, but not locally connected or arcwise connected.

Attractors

Suppose $F : D \to D$ is a topological dynamical system on a metric space D.

Definition. A compact set $N \subset D$ is called a **trapping region** for F if $F(N) \subset \mathrm{int}(N)$.

If N is a trapping region, then $N, F(N), F^2(N), \dots$ are nested compact sets and their intersection Λ is an invariant set: $F(\Lambda) \subset \Lambda$.

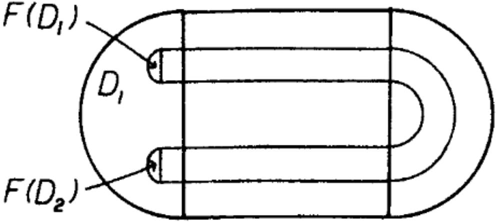
Definition. A set $\Lambda \subset D$ is called an **attractor** for F if there exists a neighborhood N of Λ such that the closure $\overline{N}$ is a trapping region for F and $\Lambda = N \cap F(N) \cap F^2(N) \cap \dots$

The attractor Λ is **transitive** if the restriction of F to Λ is a transitive map.

Examples of attractors

- The solenoid is a transitive attractor.

- Any attracting fixed point or an attracting periodic orbit is a transitive attractor.



- The horseshoe map has an attractor that is not transitive.