MATH 614

Dynamical Systems and Chaos

Lecture 18:
Stable and unstable manifolds.
Hyperbolic sets.

Hyperbolic periodic points

Suppose $F : \mathbb{R}^n \to \mathbb{R}^n$ is a differentiable map.

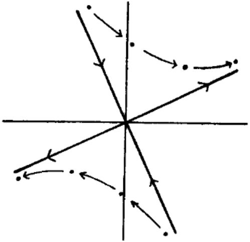
Definition. A fixed point p of the map F is **hyperbolic** if the Jacobian matrix $DF(p)$ has no eigenvalues of absolute value 1 or 0. A periodic point p of period n of the map F is **hyperbolic** if p is a hyperbolic fixed point of the map F^n.

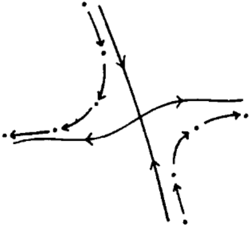
Notice that $DF^n(p) = DF(F^{n-1}(p)) \dots DF(F(p))\, DF(p)$. It follows that $DF^n(p), DF^n(F(p)), \dots, DF^n(F^{n-1}(p))$ are similar matrices. In particular, they have the same eigenvalues.

Definition. The hyperbolic periodic point p of period n is a **sink** if every eigenvalue λ of $DF^n(p)$ satisfies $0 < |\lambda| < 1$, a **source** if every eigenvalue λ satisfies $|\lambda| > 1$, and a **saddle point** otherwise.

Saddle point

The following figures show the phase portrait of a linear and a nonlinear two-dimensional maps near a saddle point.





Stable and unstable manifolds

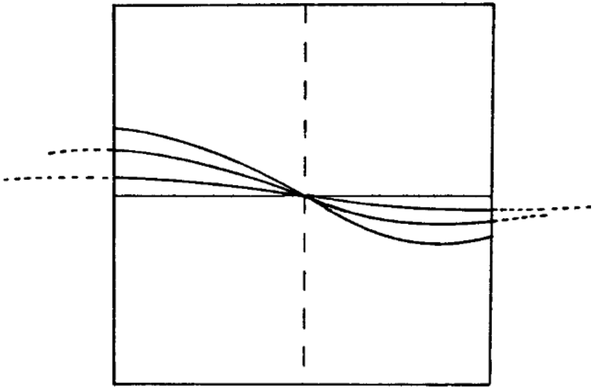
Let $F : \mathbb{R}^2 \to \mathbb{R}^2$ be a diffeomorphism and suppose p is a saddle point of F of period m.

Theorem There exists a smooth curve $\gamma : (-\varepsilon, \varepsilon) \to \mathbb{R}^2$ such that

(i) $\gamma(0) = p$;
(ii) $\gamma'(0)$ is an unstable eigenvector of $DF^m(p)$;
(iii) $F^{-1}(\gamma) \subset \gamma$;
(iv) $\|F^{-n}(\gamma(t)) - F^{-n}(p)\| \to 0$ as $n \to \infty$.
(v) $\|F^{-n}(x) - F^{-n}(p)\| < \varepsilon$ for all $n \geq 0$, then $x = \gamma(t)$ for some t.

The curve γ is called the **local unstable manifold** of F at p. The **local stable manifold** of F at p is defined as the local unstable manifold of F^{-1} at p.

Stable and unstable manifolds



Chain recurrence

Suppose X is a metric space with a distance function d.
Let $F : X \to X$ be a continuous transformation.

Definition. A point $x \in X$ is **recurrent** for the map F if for any $\varepsilon > 0$ there is an integer $n > 0$ such that $d\big(F^n(x), x\big) < \varepsilon$. The point x is **chain recurrent** for F if, for any $\varepsilon > 0$, there are points $x_0 = x, x_1, x_2, \dots, x_k = x$ and positive integers $n_1, n_2, \dots, n_k$ such that $d\big(F^{n_i}(x_{i-1}), x_i\big) < \varepsilon$ for $1 \leq i \leq k$.

A sequence $x_0, x_1, \dots, x_k$ is called an ε-**pseudo-orbit** of the map F if $d\big(F(x_{i-1}), x_i\big) < \varepsilon$ for $1 \leq i \leq k$. The point $x \in X$ is chain recurrent for F if, for any $\varepsilon > 0$, there exists an ε-pseudo-orbit $x_0, x_1, \dots, x_k$ with $x_0 = x_k = x$.

Morse-Smale diffeomorphisms

Definition. A diffeomorphism $F : X \to X$ is called **Morse-Smale** if
(i) it has only finitely many chain recurrent points,
(ii) every chain recurrent point is periodic,
(iii) every periodic point is hyperbolic,
(iv) all intersections of stable and unstable manifolds of saddle points of F are transversal.

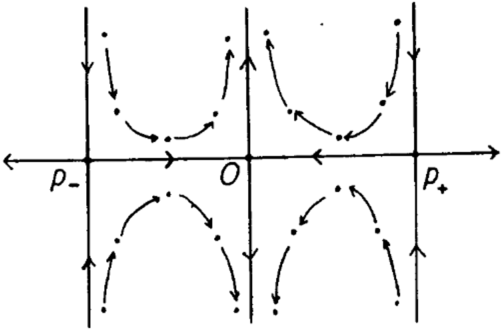
Theorem (Palis) Any Morse-Smale diffeomorphism of a compact surface is C^1-structurally stable.

Example

- $F(x, y) = (x_1, y_1)$, where $x_1 = \frac{1}{2}(x + x^3)$,

$y_1 = y \cdot \dfrac{2}{1 + 2x^2}$.

There are three fixed points: $p_+ = (1, 0)$, $p_- = (-1, 0)$ and $O = (0, 0)$. All three are saddle points.

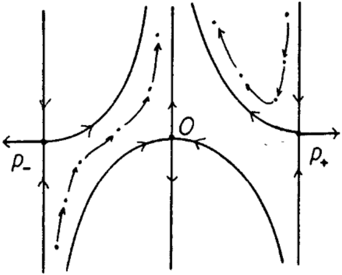


Example

- $F(x, y) = (x_1, y_1)$, where $x_1 = \frac{1}{2}(x + x^3)$,

$y_1 = y \cdot \frac{2}{1 + 2x^2} + \phi(|x|)$, where $\phi(t) > 0$ for $0 < t < 1$ and $\phi(t) = 0$ otherwise.

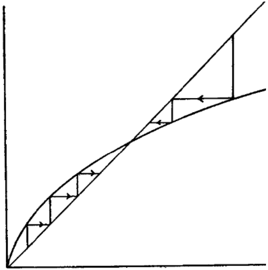
There are still three fixed points: $p_+ = (1, 0)$, $p_- = (-1, 0)$ and $O = (0, 0)$. All three are still saddle points.

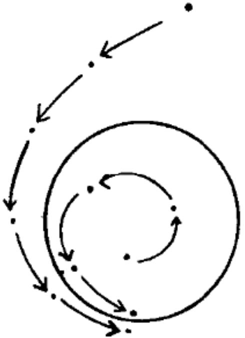


The map F is a Morse-Smale diffeomorphism.

Example

In polar coordinates (r, θ), $F(r, \theta) = (r_1, \theta_1)$, where $r_1 = 2r - r^3$, $\theta_1 = \theta + 2\pi\omega$.

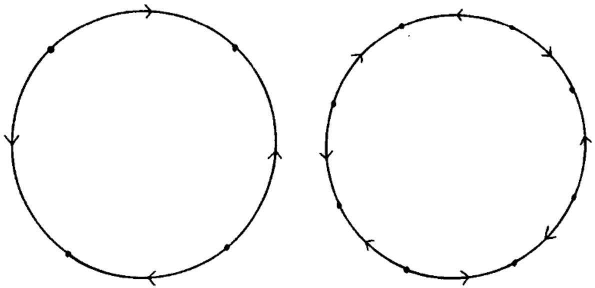




The chain recurrent points are the origin and all points of the invariant circle $r = 1$.

Example

$F(r,\theta) = (r_1, \theta_1)$, where $r_1 = 2r - r^3$,
$\theta_1 = \theta + 2\pi(p/q) + \varepsilon \sin(q\omega)$, $p, q \in \mathbb{Z}$ and $\varepsilon > 0$ is small.



The restriction of F to the invariant circle $r = 1$ is a Morse-Smale diffeomorphism of the circle. It follows that F is a Morse-Smale diffeomorphism of the plane.

Hyperbolic set

Suppose $F : D \to D$ is a diffeomorphism of a domain $D \subset \mathbb{R}^k$.

Definition. A set $\Lambda \subset D$ is called a **hyperbolic set** for F if for any $x \in \Lambda$ there exists a pair of subspaces $E^s(x), E^u(x) \subset \mathbb{R}^k$ such that

(i) $\mathbb{R}^k = E^s(x) \oplus E^u(x)$ for all $x \in \Lambda$;

(ii) $DF\big(E^s(x)\big) = E^s(F(x))$ and $DF\big(E^u(x)\big) = E^u(F(x))$ for all $x \in \Lambda$;

(iii) the subspaces $E^s(x)$ and $E^u(x)$ vary continuously with x;

(iv) there is a constant $\lambda > 1$ such that $\|DF(x)\mathbf{v}\| \geq \lambda\|\mathbf{v}\|$ for all $\mathbf{v} \in E^u(x)$ and $\|DF(x)\mathbf{v}\| \leq \lambda^{-1}\|\mathbf{v}\|$ for all $\mathbf{v} \in E^s(x)$.

Hyperbolic set

Conditions (ii) and (iv) imply that
$\|DF^n(x)\mathbf{v}\| \geq \lambda^n \|\mathbf{v}\|$ for all $\mathbf{v} \in E^u(x)$ and
$\|DF^n(x)\mathbf{v}\| \leq \lambda^{-n} \|\mathbf{v}\|$ for all $\mathbf{v} \in E^s(x)$.

Note that condition (iv) may not be preserved under changes of coordinates. We can modify it as follows:

(iv′) there are constants $c, \lambda > 1$ such that
$\|DF^n(x)\mathbf{v}\| \geq c^{-1}\lambda^n \|\mathbf{v}\|$ for all $\mathbf{v} \in E^u(x)$ and
$\|DF^n(x)\mathbf{v}\| \leq c\lambda^{-n} \|\mathbf{v}\|$ for all $\mathbf{v} \in E^s(x)$.

Stable and unstable manifolds

Let $F : \mathbb{R}^k \to \mathbb{R}^k$ be a diffeomorphism and suppose Λ is a compact invariant hyperbolic set for F. Assume that $\dim E^u(x) = 1$ for all $x \in \Lambda$ (this is automatic if $k = 2$).

Theorem There exists $\varepsilon > 0$ and, for any $x \in \Lambda$, a smooth curve $\gamma_x : (-\varepsilon, \varepsilon) \to \mathbb{R}^2$ such that

(i) $\gamma_x(0) = x$;
(ii) $\gamma_x'(0) \in E^u(x) \setminus \{\mathbf{0}\}$;
(iii) γ_x depends continuously on x;
(iv) $F(\gamma_x) \supset \gamma_{F(x)}$;
(v) $\|F^{-n}(\gamma_x(t)) - F^{-n}(x)\| \to 0$ as $n \to \infty$.

The curve γ_x is called the **local unstable manifold** of F at x. In the case $\dim E^u(x) = d > 1$, the theorem holds as well, with curves γ_x replaced by d-dimensional smooth manifolds. The **local stable manifold** of F at a point x is defined as the local unstable manifold of F^{-1} at x.

Examples of hyperbolic sets

- For any hyperbolic periodic point, the orbit is a hyperbolic set.

- For a hyperbolic toral automorphism, the entire torus is a hyperbolic set (such a map is called an Anosov map; it is C^1-structurally stable).

- For the horseshoe map, the invariant Cantor set is hyperbolic. It is an example where all chain recurrent points form a hyperbolic set (Axiom A map). Such a map is structurally stable.

Shadowing Lemma

Suppose X is a metric space with a distance function d.
Let $F : X \to X$ be a continuous transformation.

Definition. We say that a sequence $x_n, x_{n+1}, \ldots, x_m$ of elements of X is δ-**shadowed** by the orbit of a point $y \in X$ if $d\big(F^i(y), x_i\big) < \delta$ for $n \leq i \leq m$.

Recall that the sequence $x_n, x_{n+1}, \ldots, x_m$ is an ε-pseudo-orbit of the map F if $d\big(F(x_{i-1}), x_i\big) < \varepsilon$ for $n < i \leq m$.

Theorem (Bowen) Suppose F is a diffeomorphism that admits an invariant hyperbolic set Λ. Then for any $\varepsilon > 0$ there exists $\delta > 0$ such that every ε-pseudo-orbit $x_n, x_{n+1}, \ldots, x_m$ of elements of Λ is δ-shadowed by the orbit of some $y \in \Lambda$.