MATH 614

Dynamical Systems and Chaos

Lecture 22:
The Julia and Fatou sets (continued).

The Julia set

Suppose $P : U \to U$ is a holomorphic map, where U is a domain in $\mathbb{C}$, the entire plane $\mathbb{C}$, or the Riemann sphere $\overline{\mathbb{C}}$.

Definition. The **Julia set** $J(P)$ of P is the closure of the set of repelling periodic points of P.

Examples. • $L_2(z) = 2z$.
$J(L_2) = \{0\}$.

• $L_{1/2}(z) = z/2$.
$J(L_{1/2}) = \{\infty\}$ since $L_{1/2} = H \circ L_2 \circ H^{-1}$, where $H(z) = 1/z$.

• $L_{1,1}(z) = z + 1$.
$J(L_{1,1}) = \emptyset$.

• $Q_0(z) = z^2$.
$J(Q_0) = \{z \in \mathbb{C} : |z| = 1\}$.

• $Q_{-2}(z) = z^2 - 2$.
$J(Q_{-2}) = [-2, 2]$. Note that $Q_{-2}(2\cos\alpha) = 2\cos(2\alpha)$.

Normal family

Let $\mathcal{F}$ be a collection of holomorphic functions $F : U \to \mathbb{C}$ defined in a domain $U \subset \mathbb{C}$.

Definition. The collection $\mathcal{F}$ is a **normal family** in U if every sequence $F_1, F_2, \dots$ of functions from $\mathcal{F}$ has a subsequence $F_{n_1}, F_{n_2}, \dots$ $(1 \le n_1 < n_2 < \dots)$ which either
(i) converges uniformly on compact subsets of U, or
(ii) converges uniformly to ∞ on U.

The condition (i) means that there exists a function $f : U \to \mathbb{C}$ such that for any compact set $D \subset U$ we have

$$\sup_{z \in D} |F_{n_k}(z) - f(z)| \to 0 \text{ as } k \to \infty.$$

The function f is going to be continuous.

The condition (ii) means that for any $R > 0$ there exists an integer $K > 0$ such that $|F_{n_k}(z)| > R$ for all $k \ge K$ and $z \in U$.

The Fatou set

Let $\mathcal{F}$ be a collection of holomorphic functions defined in a domain $U \subset \overline{\mathbb{C}}$.

We say that the collection $\mathcal{F}$ is **normal at** a finite point $z \in U$ if it is a normal family in some neighborhood of z. In the case $\infty \in U$, we say $\mathcal{F}$ is **normal at infinity** if the collection of functions $G(z) = F(1/z)$, $F \in \mathcal{F}$ is normal at 0.

Definition. The **Fatou set** $S(P)$ of a holomorphic map $P : U \to U$ is the set of all points $z \in U$ such that the family of iterates $\{P^n\}_{n \geq 1}$ is normal at z.

By definition, the Fatou set is open.

Let $\mathcal{F}$ be a collection of holomorphic functions defined in a domain $U \subset \overline{\mathbb{C}}$.

Theorem (Arzelà-Ascoli) Suppose the functions in $\mathcal{F}$ are uniformly bounded and their derivatives are uniformly bounded.

Then any sequence $F_1, F_2, \ldots$ of functions from $\mathcal{F}$ has a subsequence $F_{n_1}, F_{n_2}, \ldots$ which converges uniformly on compact subsets of U.

Corollary If the iterates of a holomorphic transformation P and their derivatives are uniformly bounded in a neighborhood of a point z, then $z \in S(P)$.

Theorem (Weierstrass) Let $F_1, F_2, \ldots$ be holomorphic functions in a domain U. Assume that the sequence $F_1, F_2, \ldots$ converges uniformly on compact subsets of U.

Then the limit function F is holomorphic in U and, moreover, the sequence of derivatives $F_1', F_2', \ldots$ converges to F' uniformly on compact subsets of U.

Corollary Let $z \in \mathbb{C}$. Assume that there exists a sequence of iterates $P^{n_1}, P^{n_2}, \ldots$ such that $P^{n_k}(z) \not\to \infty$ while $(P^{n_k})'(z) \to \infty$ as $k \to \infty$. Then $z \notin S(P)$.

The Fatou set and periodic points

Proposition 1 Attracting periodic points of P belong to $S(P)$.

Proof: Suppose z_0 is an attracting periodic point of period n. Then $P^n(z_0) = z_0$ and $|P^n(z) - z_0| \leq \mu |z - z_0|$ for some $0 < \mu < 1$ and all z close enough to z_0. Hence there is a neighborhood D of z_0 such that the functions $P^n, P^{2n}, P^{3n}, \dots$ converge to the constant z_0 uniformly on D. Then for any integer $k \geq 1$ the functions $P^{n+k}, P^{2n+k}, P^{3n+k}, \dots$ converge to the constant $P^k(z_0)$ uniformly on D.

It follows that the sequence $P, P^2, P^3, \dots$ is normal at z_0.

The Fatou set and periodic points

Proposition 2 Repelling periodic points of P do not belong to $S(P)$.

Proof: Suppose z is a repelling periodic point of period n. Then $P^n(z) = z$ and $|(P^n)'(z)| > 1$. For any integer $k \geq 1$ we have $P^{nk}(z) = z$ and $(P^{nk})'(z) = \big((P^n)'(z)\big)^k$. As a consequence, $P^{nk}(z) \not\to \infty$ while $(P^{nk})'(z) \to \infty$ as $k \to \infty$. By the above, $z \notin S(P)$.

Corollary The Julia set and the Fatou set of P are disjoint.

The Fatou set and periodic points

Neutral periodic points of P may or may not belong to $S(P)$.

Examples. • $P(z) = e^{i\alpha} z$, where $\alpha \in \mathbb{R}$.

The neutral fixed point 0 belongs to the Fatou set $S(P)$. Indeed, $P^n(z) = e^{in\alpha} z$ and $(P^n)'(z) = e^{in\alpha}$ for all $z \in \mathbb{C}$ and $n = 1, 2, \dots$ Therefore all iterates of P and their derivatives are uniformly bounded in a neighborhood of 0.

• $P(z) = z + z^2$.

The neutral fixed point 0 does not belong to the Fatou set $S(P)$. Indeed, for any ε, $0 < \varepsilon < 1$, we have $P^n(-\varepsilon) \to 0$ as $n \to \infty$ while $P^n(\varepsilon) \to \infty$ as $n \to \infty$. It follows that the iterates of P are not normal at 0.

Invariance

Proposition 1 The Julia set of a holomorphic map $P : U \to U$ is invariant: $P(J(P)) \subset J(P)$.

Proof: Let $z \in J(P)$. There are repelling periodic points $z_1, z_2, \dots$ of P such that $z_n \to z$ as $n \to \infty$. By continuity, $P(z_n) \to P(z)$ as $n \to \infty$. Clearly, $P(z_1), P(z_2), \dots$ are also repelling periodic points of P.

Proposition 2 $P(J(P)) = J(P)$.

Proof: Let $z \in J(P)$ and $z_n \to z$ as $n \to \infty$, where $z_1, z_2, \dots$ are repelling periodic points of P. Then there are repelling periodic points $w_1, w_2, \dots$ such that $P(w_n) = z_n$. The sequence $w_1, w_2, \dots$ is bounded, hence there is a converging subsequence: $w_{n_k} \to w$ as $k \to \infty$. Then $z_{n_k} = P(w_{n_k}) \to P(w)$. We have $w \in J(P)$ and $P(w) = z$.

Invariance

Proposition 3 The Fatou set of P is completely invariant under this map: $P(S(P)) \subset S(P)$ and $P^{-1}(S(P)) \subset S(P)$.

Proof: Suppose $P(w) = z$. We have to show that the family $P, P^2, P^3, \ldots$ is normal at z if and only if it is normal at w.

Indeed, let f be a nonconstant holomorphic function such that $f(w) = z$. Then $P, P^2, P^3, \ldots$ is normal at z if and only if the family $P \circ f, P^2 \circ f, P^3 \circ f, \ldots$ is normal at w.

The Fatou components

The Fatou set $S(P)$ of a nonconstant holomorphic map $P : U \to U$ is open. Connected components of this set are called the **Fatou components** of P.

Proposition 1 For any Fatou component D of P, the image $P(D)$ is also a Fatou component of P.

Proposition 2 For any Fatou component D of a rational map P there exist integers $k \geq 0$ and $n \geq 1$ such that the Fatou component $P^k(D)$ is invariant under P^n.

The Fatou components

There are 4 types of invariant Fatou components for rational maps:

- **immediate basin of attraction** of an attracting fixed point lying inside the component;
- **attracting petal** of a neutral fixed point lying on the boundary of the component;
- **Siegel disc**: the restriction of the map to the component is holomorphically conjugate to a rotation of the disc;
- **Herman ring**: the restriction of the map to the component is holomorphically conjugate to a rotation of the annulus.

Montel's Theorem

Theorem (Montel) Suppose $\mathcal{F}$ is a family of holomorphic functions defined on a domain $U \subset \mathbb{C}$. If the functions from $\mathcal{F}$ do not assume two values $a, b \in \mathbb{C}$, then $\mathcal{F}$ is a normal family in U.

Corollary 1 If $P : U \to U$ is a holomorphic map, where $U \subset \mathbb{C}$ and $\mathbb{C} \setminus U$ contains at least two points, then $S(P) = U$ and $J(P) = \emptyset$.

Corollary 2 Suppose $z \in J(P)$ and W is a neighborhood of z. Then $\bigcup_{n=1}^{\infty} P^n(W)$ is either $\mathbb{C}$ or $\mathbb{C}$ minus one point.

Theorem If the Julia set $J(P)$ consists of more than one repelling orbit, then the map P is chaotic on $J(P)$.

Proof: **(i)** Periodic points of P are dense in $J(P)$. This holds by definition of $J(P)$.

(ii) Topological transitivity: for any nonempty open sets $U_1, U_2 \subset J(P)$ there exists $n \geq 1$ such that $P^n(U_1) \cap U_2 \neq \emptyset$.
Here $U_1 = W_1 \cap J(P)$, $U_2 = W_2 \cap J(P)$, where W_1, W_2 are open sets in $\mathbb{C}$.
We know that $\bigcup_{n \geq 1} P^n(W_1)$ is $\mathbb{C}$ or $\mathbb{C}$ minus one point. It follows that $P^n(W_1) \cap U_2 \neq \emptyset$ for some n. But $P^n(W_1) \cap U_2 = P^n(U_1) \cap U_2$.

(iii) Sensitive dependence on initial conditions:
there exists $\beta > 0$ such that for any $z_0 \in J(P)$ and any neighborhood U of z_0 (in $J(P)$) we have
$|P^n(z) - P^n(z_0)| \geq \beta$ for some $n \geq 1$ and $z \in U$.

The Julia set contains two different repelling periodic orbits: $z_1, z_2, \ldots, z_m$ and $w_1, w_2, \ldots, w_k$. Choose $\beta > 0$ so that $|z_j - w_l| > 2\beta$ for all j and l.
We know that $\bigcup_{n \geq 1} P^n(U) = J(P)$. Therefore we can find $z, w \in U$ such that $P^{n_1}(z) = z_1$ and $P^{n_2}(w) = w_1$ for some $n_1, n_2 \geq 1$. Now take any $n \geq \max(n_1, n_2)$. Then $P^n(z)$ is in the cycle $z_1, z_2, \ldots, z_m$ while $P^n(w)$ is in the cycle $w_1, w_2, \ldots, w_k$.
In particular, $|P^n(z) - P^n(w)| > 2\beta$. It follows that $|P^n(z) - P^n(z_0)| > \beta$ or $|P^n(w) - P^n(z_0)| > \beta$.