MATH 614

Dynamical Systems and Chaos

Lecture 23:
The filled Julia set.
The Mandelbrot set.

The Julia and Fatou sets

Suppose $P : U \to U$ is a holomorphic map, where U is a domain in $\mathbb{C}$, the entire plane $\mathbb{C}$, or the Riemann sphere $\overline{\mathbb{C}}$.

Definition. The **Julia set** $J(P)$ of P is the closure of the set of repelling periodic points of P. The **Fatou set** $S(P)$ of P is the set of all points $z \in U$ such that the family of iterates $\{P^n\}_{n\geq 1}$ is normal at z.

- The Julia set is closed, the Fatou set is open.
- The Julia and Fatou sets are disjoint.
- Attracting periodic points of P belong to $S(P)$.
- $P(J(P)) = J(P)$.
- $P(S(P)) \subset S(P)$ and $P^{-1}(S(P)) \subset S(P)$.
- If $U \subset \mathbb{C}$ and $\mathbb{C} \setminus U$ contains at least two points, then $S(P) = U$ and $J(P) = \emptyset$.
- If the Julia set $J(P)$ is more than one repelling orbit, then the map P is chaotic on $J(P)$.

Montel's Theorem

Theorem (Montel) Suppose $\mathcal{F}$ is a family of holomorphic functions $F : U \to \mathbb{C}$ defined on a domain $U \subset \mathbb{C}$. If the functions from $\mathcal{F}$ do not assume two values $a, b \in \mathbb{C}$, then $\mathcal{F}$ is a normal family in U.

Corollary Suppose $P : \mathbb{C} \to \mathbb{C}$ is a holomorphic function, $z \in J(P)$, and W is a neighborhood of z. Then for any integer $k \geq 1$, the union $\bigcup_{n=k}^{\infty} P^n(W)$ is either $\mathbb{C}$ or $\mathbb{C}$ minus one point.

More properties of the Julia and Fatou sets

Proposition 1 If the Fatou set is not empty, then the Julia set has empty interior.

Proposition 2 There exists a rational function P such that $J(P) = \overline{\mathbb{C}}$ and $S(P) = \emptyset$.

Proposition 3 If the Julia set is more than one repelling orbit, then it has no isolated points.

Theorem The union of the Julia and Fatou sets of P is the entire domain of P.

The Fatou components

The Fatou set $S(P)$ of a nonconstant holomorphic map $P : U \to U$ is open. Connected components of this set are called the **Fatou components** of P.

- For any Fatou component D of P, the image $P(D)$ is also a Fatou component of P.

- For any Fatou component D of a rational function P there exist integers $k \geq 0$ and $n \geq 1$ such that the Fatou component $P^k(D)$ is invariant under P^n.

- Some transcendental functions P admit a Fatou component D that is a **wandering domain**, i.e., $D, P(D), P^2(D), \dots$ are disjoint sets.

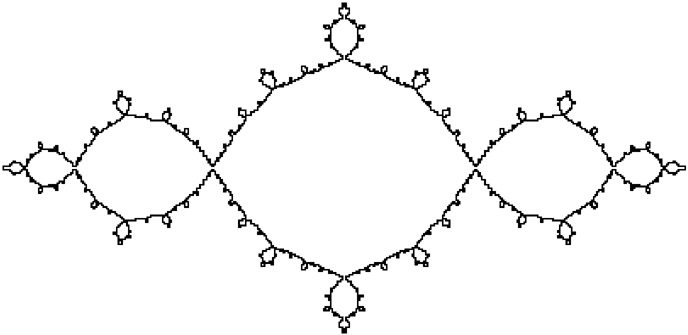
The Fatou components

There are 5 types of invariant Fatou components for a holomorphic map $P : U \to U$:

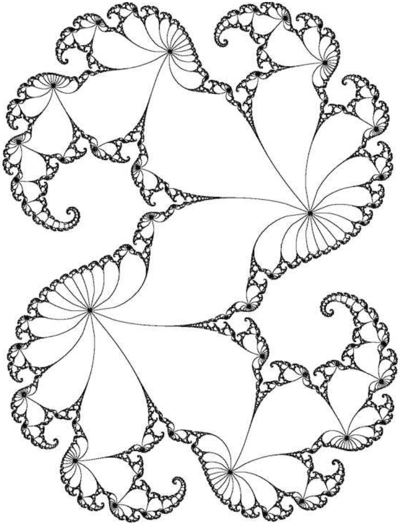
- **immediate basin of attraction** of an attracting fixed point lying inside the component;
- **attracting petal** of a neutral fixed point lying on the boundary of the component;
- **Siegel disc**: the restriction of P to the component is holomorphically conjugate to a rotation of the disc;
- **Herman ring**: the restriction of P to the component is holomorphically conjugate to a rotation of the annulus;
- **Baker domain**: the iterates of P converge (uniformly on compact subsets of the component) to a constant $z_0 \notin U$ that is an essential singularity of P.

The Baker domains cannot occur for a rational function P.
The Herman rings cannot occur for functions $P : \mathbb{C} \to \mathbb{C}$.

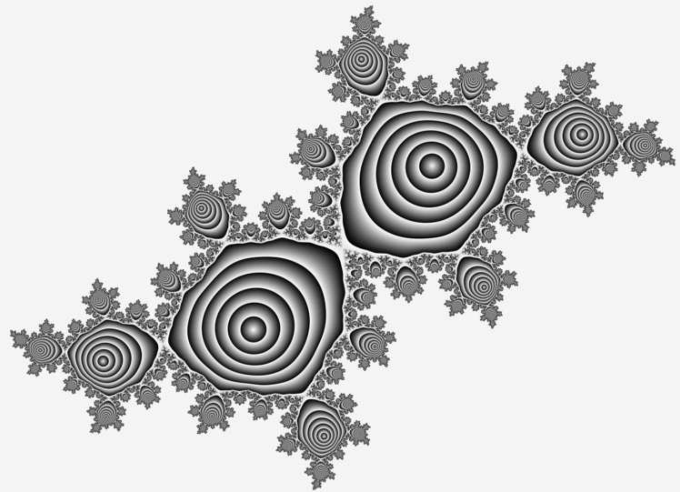
Basin of attraction



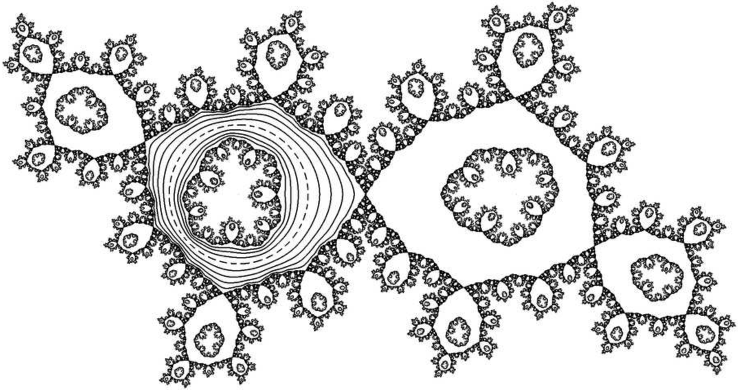
Attracting petals



Siegel discs



Herman rings



Polynomial maps

From now on, we assume that P is a polynomial map with $\deg P \geq 2$:

$$P(z) = a_n z^n + a_{n-1} z^{n-1} + \cdots + a_1 z + a_0,$$

where $a_n \neq 0$, $n \geq 2$. We consider P as a transformation of $\overline{\mathbb{C}}$.

Proposition The point at infinity is a super-attracting fixed point of P.

Proof: Clearly, $P(\infty) = \infty$. To find the derivative $P'(\infty)$, we need to compute the derivative $R'(0)$ of a rational function $R(z) = 1/P(1/z)$. Since $P(z) = a_n z^n + a_{n-1} z^{n-1} + \cdots + a_1 z + a_0$, it follows that $R(z) = z^n/(a_n + a_{n-1} z + \cdots + a_1 z^{n-1} + a_0 z^n)$. Since $a_n \neq 0$ and $n \geq 2$, we obtain that $R'(0) = 0$.

The filled Julia set

Definition. The **filled Julia set** of the polynomial P, denoted $K(P)$, is the set of all points $z \in \mathbb{C}$ such that the orbit $z, P(z), P^2(z), \ldots$ is bounded.

Proposition 1 The complement of $K(P)$ consists of points whose orbits escape to infinity.

Proposition 2 There is $R_0 > 0$ such that the set $\{z \in \mathbb{C} : |z| > R_0\}$ is contained in the Fatou set.

Proposition 3 The Julia set and the filled Julia set are bounded.

Proposition 4 The Julia set is contained in the filled Julia set.

More properties of the filled Julia set

- The filled Julia set is completely invariant: $P(K(P)) \subset K(P)$ and $P^{-1}(K(P)) \subset K(P)$.

- The complement of the filled Julia set is contained in the Fatou set.

- The filled Julia set is closed.

- The filled Julia set is nonempty.

- The interior of the filled Julia set is contained in the Fatou set.

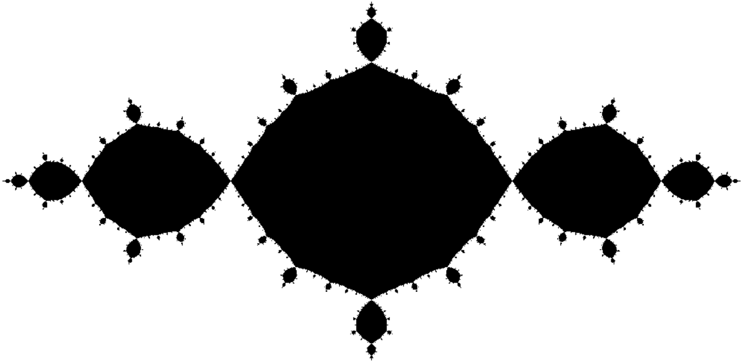
More properties of the filled Julia set

Proposition The boundary of the filled Julia set is disjoint from the Fatou set.

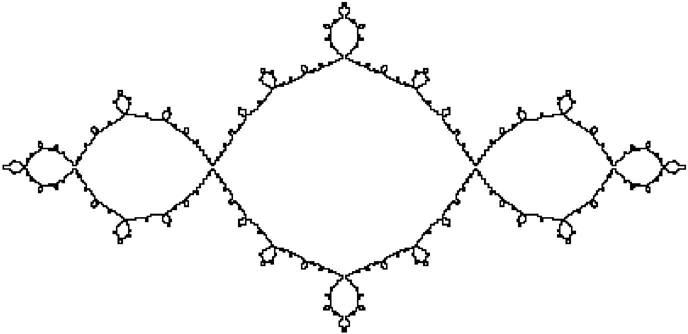
Proof: Suppose $z \in \partial K(P)$ and U is an arbitrary neighborhood of z. Then there are points $z_1, z_2 \in U$ such that $z_1 \in K(P)$ while $z_2 \notin K(P)$. We have $|P^n(z_1)| < R < \infty$ while $P^n(z_2) \to \infty$ as $n \to \infty$. It follows that the family $P, P^2, P^3, \dots$ is not normal in U.

Corollary The boundary of the filled Julia set is the complement of the Fatou set.

Theorem The Julia set is the boundary of the filled Julia set.



A filled Julia set $(z \mapsto z^2 - 1)$



A Julia set $(z \mapsto z^2 - 1)$

Quadratic family

The quadratic family $Q_c : \mathbb{C} \to \mathbb{C}$, $c \in \mathbb{C}$,
$Q_c(z) = z^2 + c$.

The dynamics of Q_c depends on properties of the post-critical orbit $0, Q_c(0), Q_c^2(0), \dots$.

Theorem (Fundamental Dichotomy)
For any $c \in \mathbb{C}$,

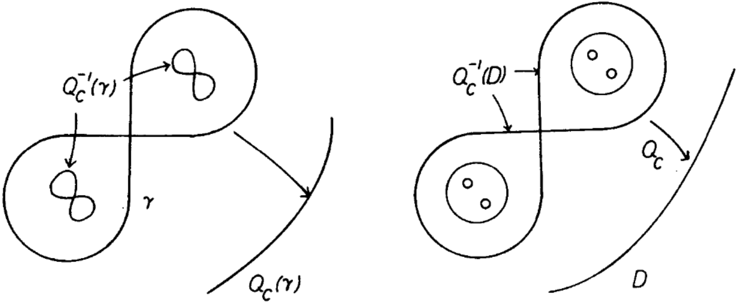
either the post-critical orbit $0, Q_c(0), Q_c^2(0), \dots$ escapes to ∞, in which case the Julia set $J(Q_c)$ is a Cantor set,

or the post-critical orbit is bounded, in which case the Julia set $J(Q_c)$ is connected.

Lemma If $|z| > \max(|c|, 2)$ then $|Q_c^n(z)| \to \infty$ as $n \to \infty$.

Suppose γ_0 is a simple closed curve in $\mathbb{C}$ and D_0 is the domain bounded by γ_0. Let $\gamma_1 = Q_c^{-1}(\gamma_0)$ and $D_1 = Q_c^{-1}(D_0)$. Then γ_1 is the boundary of D_1.

If $c \in D_0$ then γ_1 is a simple closed curve. If $c \in \gamma_0$ then γ_1 is a "figure eight". Otherwise γ_1 consists of two curves.



If D_0 is a large disk then $D_1 = Q_c^{-1}(D_0)$ is contained in D_0. Hence $D_0 \supset D_1 \supset D_2 \supset \dots$, where $D_{n+1} = Q_c^{-1}(D_n)$, $n \geq 0$.

Note that $K(Q_c) = \bigcap_{n=0}^{\infty} D_n$.

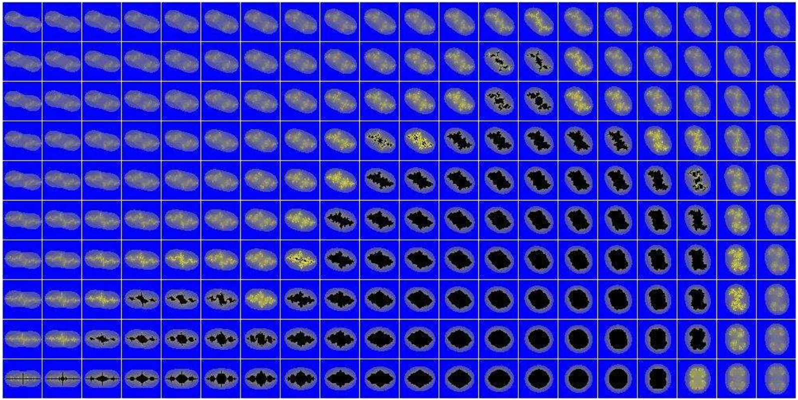
Now either $c \in D_n$ for all n, in which case each D_n is a simply connected domain so that $K(Q_c)$ and $J(Q_c)$ are also connected, or else $c \in D_n \setminus D_{n+1}$ for some n, in which case D_{n+k} has 2^{k-1} connected components, $k = 0, 1, 2, \dots$, so that $K(Q_c) = J(Q_c)$ is a Cantor set.

The Mandelbrot set

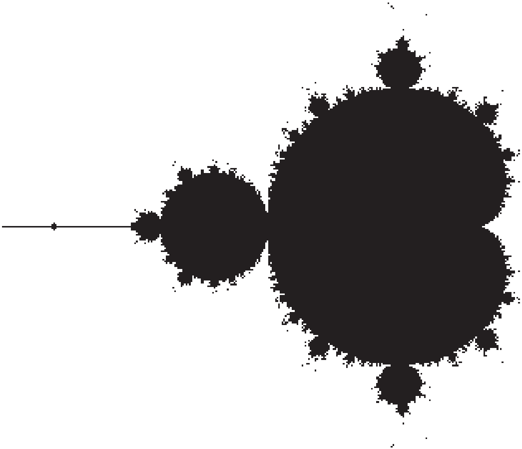
Definition. The **Mandelbrot set** $\mathcal{M}$ is the set of all $c \in \mathbb{C}$ such that $|Q_c^n(0)| \not\to \infty$ as $n \to \infty$.

$c \in \mathcal{M}$ if and only if the Julia set $J(Q_c)$ is connected.

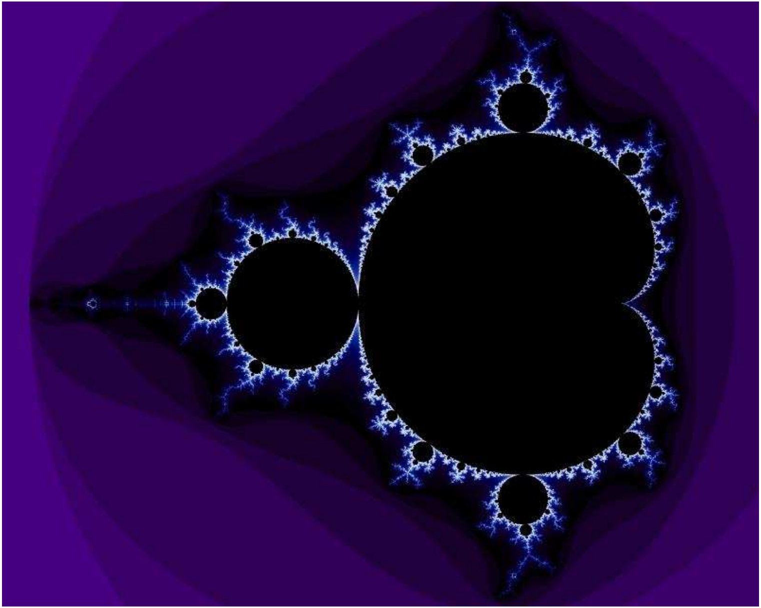
The Mandelbrot set $\mathcal{M}$ is the bifurcation diagram for the quadratic family.

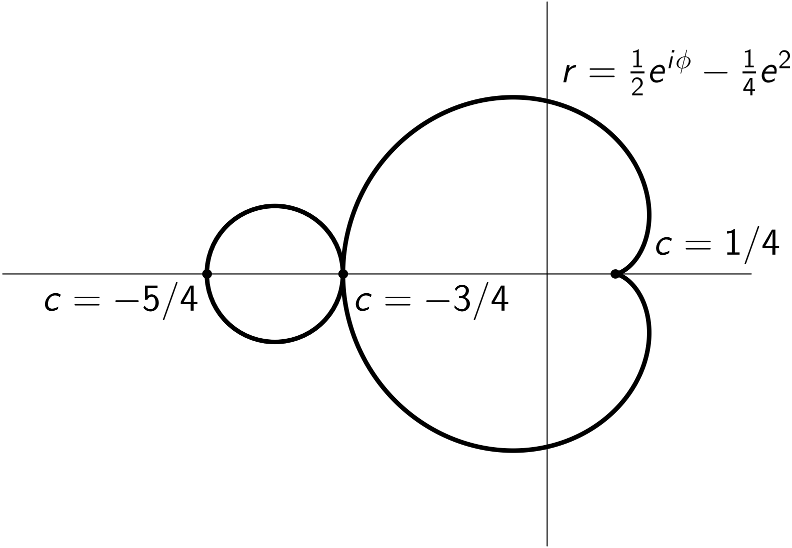


The filled Julia sets of polynomials $Q_c(z) = z^2 + c$



The Mandelbrot set



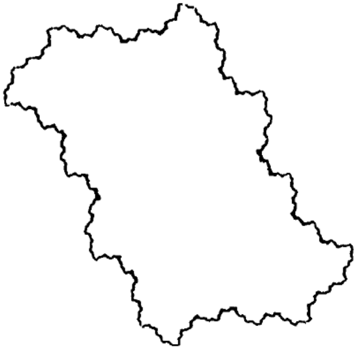


The main cardioid and the period 2 bulb

Theorem 1 For any c within the main cardioid, the Julia set $J(Q_c)$ is a simple closed curve.

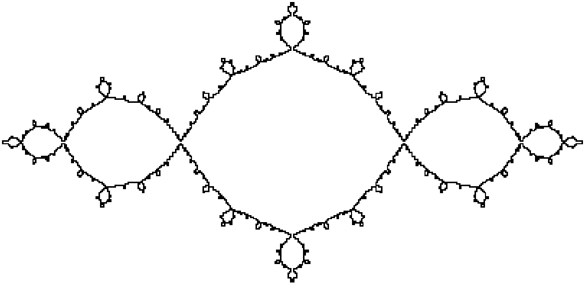
The region bounded by this curve is the basin of attraction of an attracting fixed point.

If $c \notin \mathbb{R}$ then $J(Q_c)$ contains no smooth arc.



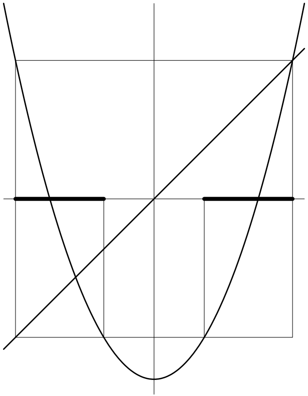
$J(Q_c)$, $c = i/2$

Theorem 2 For any c within the period 2 bulb, $J(Q_c)$ is the closure of countably many simple closed curves. The interior of $K(Q_c)$ has countably many connected components and coincides with the basin of attraction of an attracting periodic orbit of period 2.

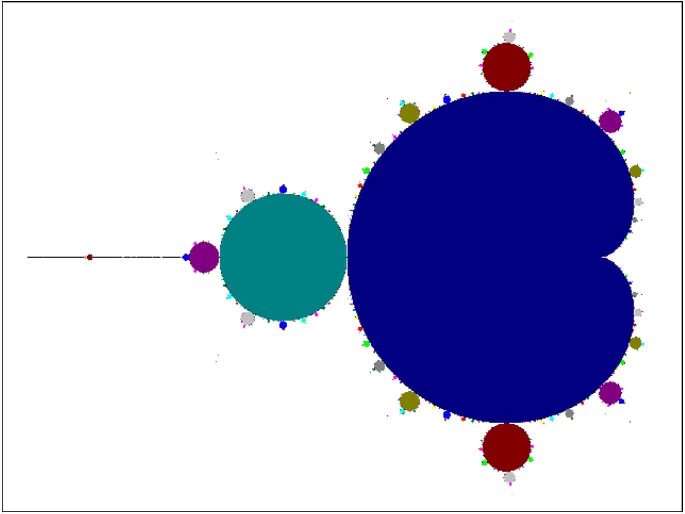


$J(Q_c),\ c = -1$

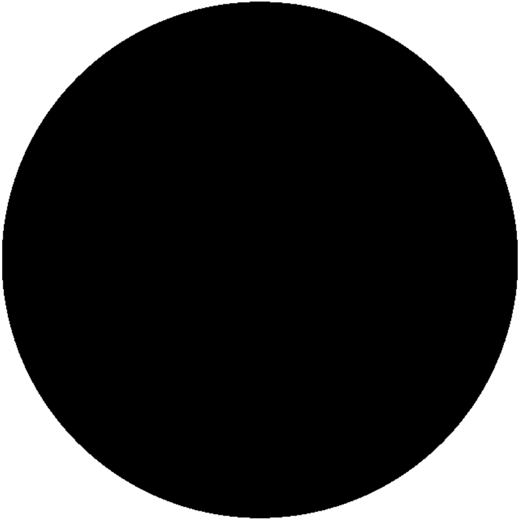
Theorem 3 For any $c \notin \mathcal{M}$, the Julia set $J(Q_c)$ is a Cantor set and the restriction of Q_c to $J(Q_c)$ is conjugate to the one-sided shift on two letters.



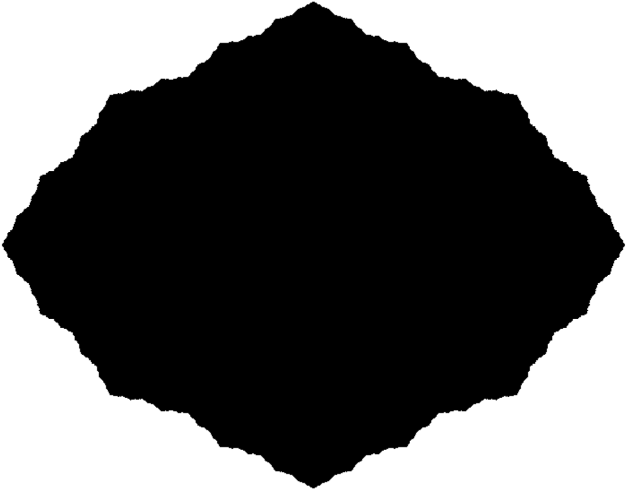
Q_c, $c < -2$. $J(Q_c) = K(Q_c)$ is a Cantor set.



Period bulbs of the Mandelbrot set

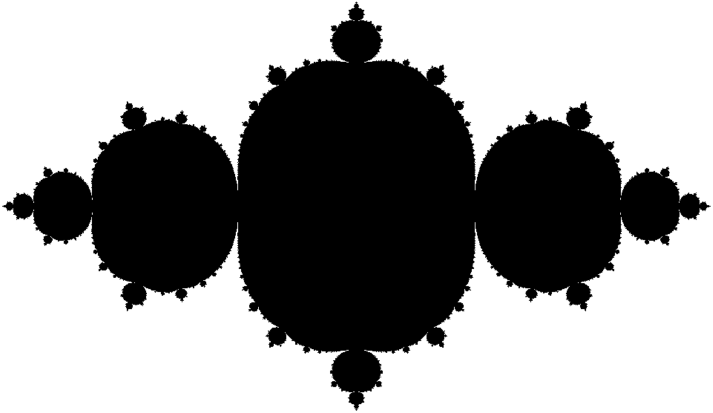


$K(Q_c),\ c = 0$

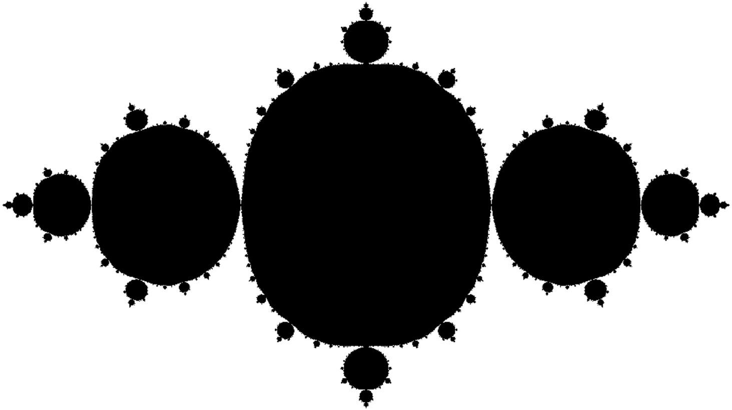


$K(Q_c),\ c = -0.3$

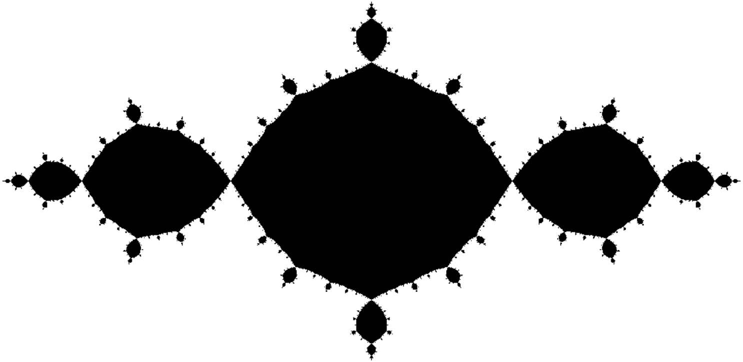
$K(Q_c),\ c = -0.6$



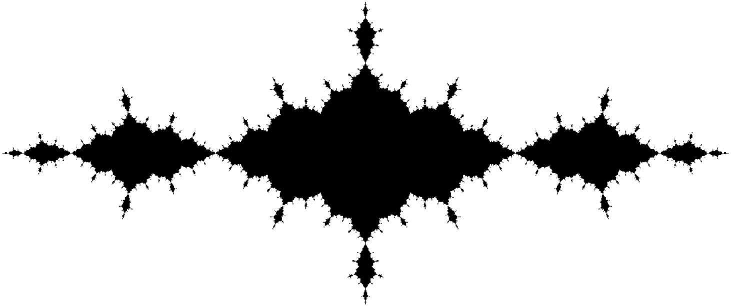
$K(Q_c),\ c = -0.75$



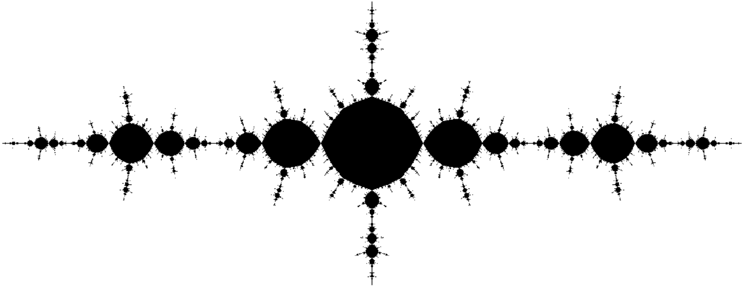
$K(Q_c),\ c = -0.8$



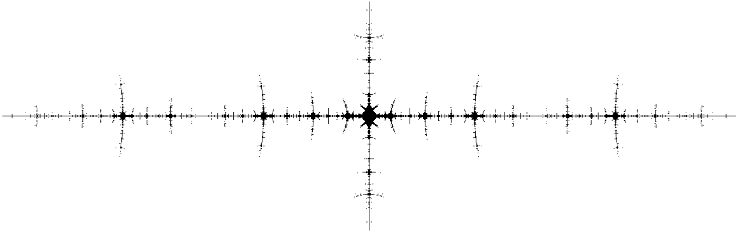
$K(Q_c),\ c = -1$



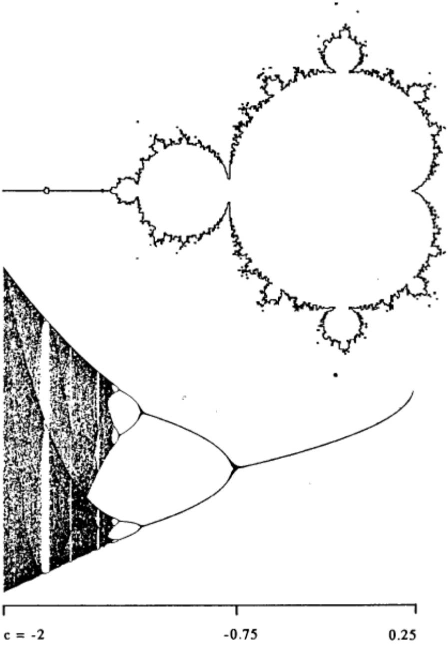
$K(Q_c),\ c = -1.2$



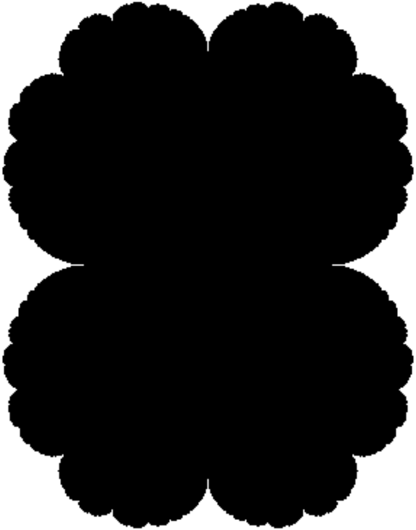
$K(Q_c), \; c = -1.3$



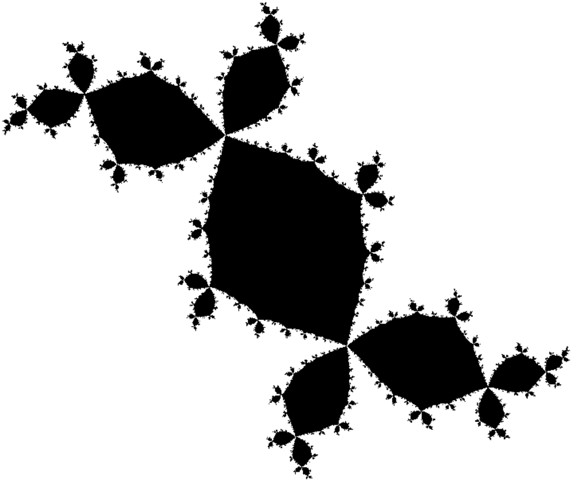
$K(Q_c),\ c = -1.5$


c = -2
-0.75
0.25

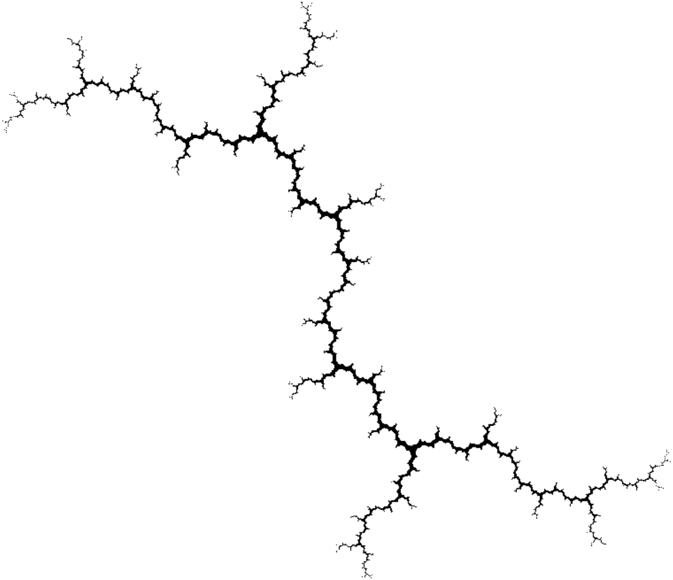
The filled Julia set of $z \mapsto z^2 + 0.2$



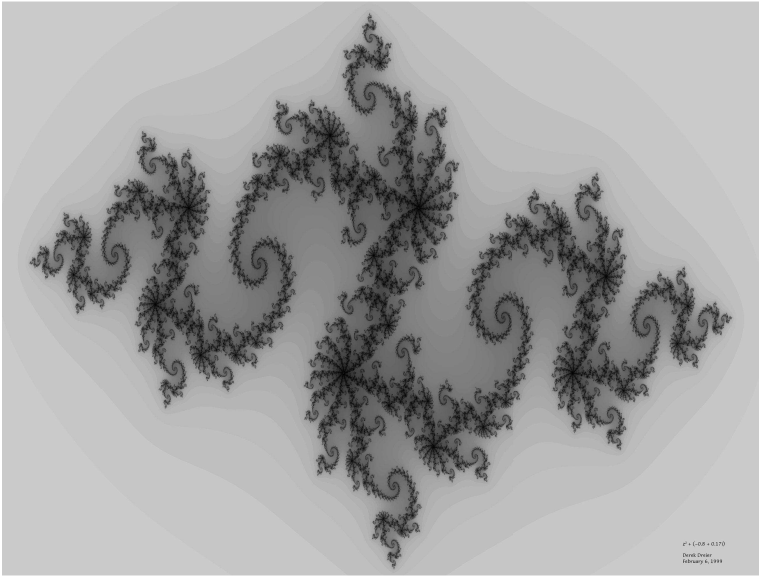
The filled Julia set of $z \mapsto z^2 + 0.25$



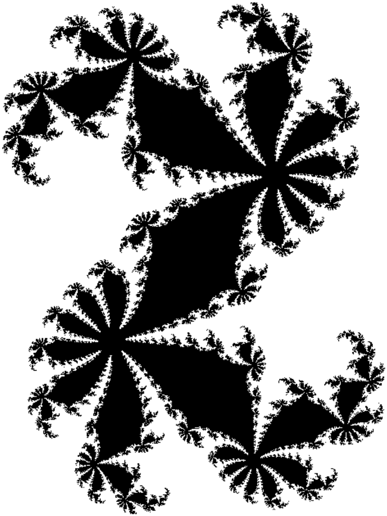
The filled Julia set of $z \mapsto z^2 - 0.122 + 0.745i$
("Rabbit")



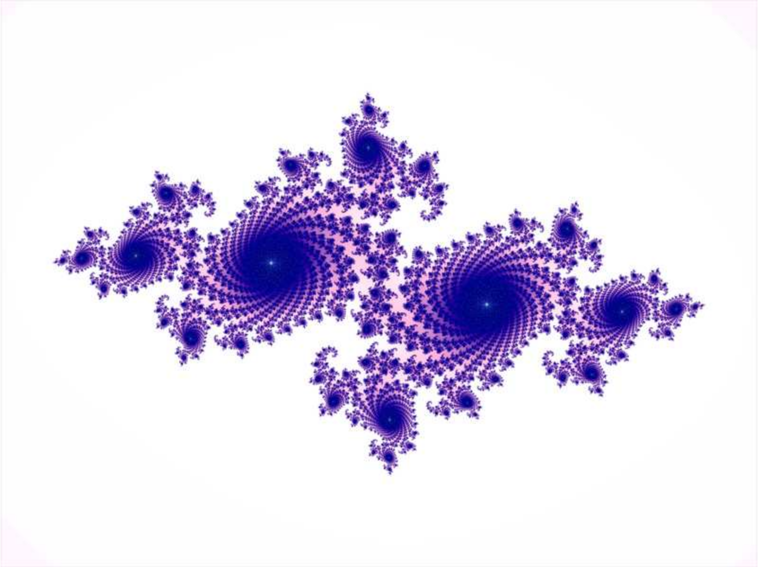
The Julia set of $z \mapsto z^2 + i$ ("Dendrite")



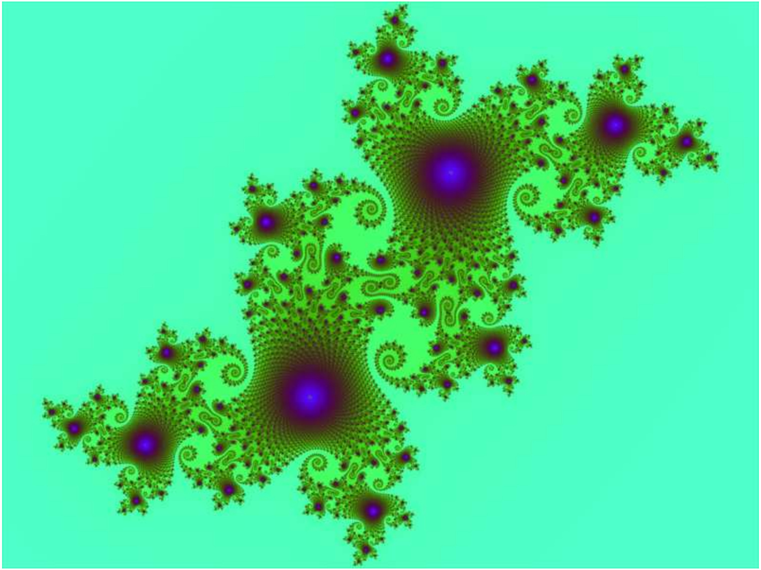
The Julia set of $z \mapsto z^2 - 0.8 + 0.17i$ ("Spiders")



“Dragon”



“Ice”



“Indigo”