MATH 614

Dynamical Systems and Chaos

Lecture 24: Invariant measure. Recurrence.

Ergodic theory

Topological dynamics is the study of continuous transformations.

Smooth dynamics is the study of smooth transformations.

Holomorphic dynamics is the study of holomorphic transformations.

Ergodic theory (a.k.a. metric theory of dynamical systems) is the study of **measure-preserving** transformations.

The **measure** is an abstract concept that generalizes the notions of length, area, and volume.

Examples

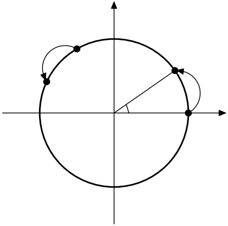
- Bijective self-map $F : X \to X$.

Any set $E \subset X$ is mapped onto a set with the same number of elements.

- Translation of the real line.

$F : \mathbb{R} \to \mathbb{R}$, $F(x) = x + x_0$. Any interval is mapped onto an interval of the same length.

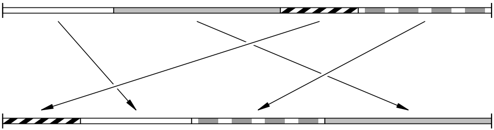
- Rotation of the circle.



Any arc is mapped onto an arc of the same length.

Non-continuous example

- Interval exchange transformation.



An **interval exchange transformation** $F : I \to I$ of an interval I is defined by cutting the interval into several subintervals and then rearranging them by translation.
The image of any subinterval $I_0 \subset I$ consists of one or several intervals whose total length equals the length of I_0.

Note that the transformation F is not well defined at the cutting points. Consequently, the orbit under F is not defined for a finite or countable set of points which may be dense in I. However this is not a concern as in ergodic theory sets of zero measure can be neglected.

- Motion of the Euclidean plane.

Any domain is mapped onto a domain of the same area.

- Linear transformation $L : \mathbb{R}^2 \to \mathbb{R}^2$.

$L(\mathbf{x}) = A\mathbf{x}$, where A is a 2×2 matrix. The image of any domain of area α has area $\alpha\,|\det A|$. In the case $\det A = \pm 1$, the map L is area-preserving.

- Translation of the torus.

$F : \mathbb{T}^2 \to \mathbb{T}^2$, $F(\mathbf{x}) = \mathbf{x} + \mathbf{x}_0$. This is the quotient of a translation of the Euclidean plane under the natural projection $\pi : \mathbb{R}^2 \to \mathbb{T}^2$.

- Toral automorphism.

$F : \mathbb{T}^2 \to \mathbb{T}^2$ $(\mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2)$, $F(\mathbf{x}) = A\mathbf{x}$, where A is a 2×2 matrix with integer entries and $\det A = \pm 1$. This is the quotient of an area-preserving linear map under the natural projection $\pi : \mathbb{R}^2 \to \mathbb{T}^2$.

Example with continuous time

- Area-preserving flow.

Consider an autonomous system of two ordinary differential equations of the first order

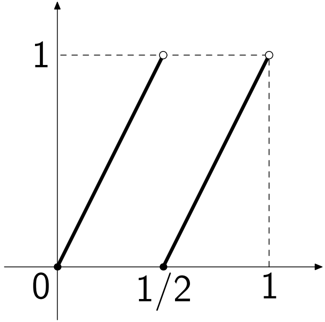
$$\begin{cases} \dot{x} = g_1(x,y), \\ \dot{y} = g_2(x,y), \end{cases}$$

where g_1, g_2 are differentiable functions defined in a domain $D \subset \mathbb{R}^2$. In vector form, $\dot{\mathbf{v}} = G(\mathbf{v})$, where $G : D \to \mathbb{R}^2$ is a vector field. Assume that for any $\mathbf{x} \in D$ the initial value problem $\dot{\mathbf{v}} = G(\mathbf{v})$, $\mathbf{v}(0) = \mathbf{x}$ has a unique solution $\mathbf{v}_{\mathbf{x}}(t)$, $t \in \mathbb{R}$. Then the system of ODEs gives rise to a dynamical system with continuous time $F^t : D \to D$, $t \in \mathbb{R}$ defined by $F^t(\mathbf{x}) = \mathbf{v}_{\mathbf{x}}(t)$ for all $\mathbf{x} \in D$ and $t \in \mathbb{R}$. In the case G is linear, $G(\mathbf{v}) = A\mathbf{v}$, the flow is also linear, $F^t(\mathbf{x}) = e^{tA}\mathbf{x}$.

The flow $\{F^t\}$ is area-preserving if and only if $\nabla \cdot G = \partial g_1/\partial x + \partial g_2/\partial y = 0$ in D.

Non-invertible example

- Doubling map $F: S^1 \to S^1$.



If $S^1 = \mathbb{R}/\mathbb{Z}$, then $F(x) = 2x$ for all $x \in S^1$. For any arc $\gamma = (\omega_1, \omega_2)$, $0 \le \omega_1 < \omega_2 \le 1$, of length $\alpha = \omega_2 - \omega_1$ the image $F(\gamma)$ is an arc of length 2α or the entire circle. However the preimage $F^{-1}(\gamma)$ consists of two disjoint arcs $(\frac{1}{2}\omega_1, \frac{1}{2}\omega_2)$ and $(\frac{1}{2}\omega_1 + \frac{1}{2}, \frac{1}{2}\omega_2 + \frac{1}{2})$ of length $\alpha/2$ so that $F^{-1}(\gamma)$ has the same length measure as γ.

Measure-preserving transformation

Definition. A **measured space** is a triple $(X, \mathcal{B}, \mu)$, where X is a set, $\mathcal{B}$ is a collection of subsets of X, and μ is a function $\mu : \mathcal{B} \to [0, \infty]$. Elements of $\mathcal{B}$ are referred to as **measurable sets**. The function μ is called the **measure** on X.

A mapping $T : X \to X$ is called **measurable** if preimage of any measurable set under T is also measurable: $E \in \mathcal{B} \implies T^{-1}(E) \in \mathcal{B}$.

A measurable mapping $T : X \to X$ is called **measure-preserving** if for any $E \in \mathcal{B}$ one has $\mu(T^{-1}(E)) = \mu(E)$.

Algebra of sets

Definition. A collection $\mathcal{B}$ of subsets of a set X is called an **algebra** of sets if $\mathcal{B}$ is closed under taking unions $B_1 \cup B_2$, intersections $B_1 \cap B_2$, complements $X \setminus B$, and if $\mathcal{B}$ contains the empty set and the entire set X.

The algebra $\mathcal{B}$ is also closed under taking finite unions $B_1 \cup B_2 \cup \cdots \cup B_n$, finite intersections $B_1 \cap B_2 \cap \cdots \cap B_n$, set differences $B_1 \setminus B_2 = B_1 \cap (X \setminus B_2)$, and symmetric differences $B_1 \triangle B_2 = (B_1 \setminus B_2) \cup (B_2 \setminus B_1)$.

For any subset $B \subset X$ let $\chi_B : X \to \{0, 1\}$ denote the characteristic function of B: $\chi_B(x) = 1$ if $x \in B$ and $\chi_B(x) = 0$ otherwise. Then $\chi_X = 1$, $\chi_\emptyset = 0$, $\chi_{B_1 \cap B_2} = \chi_{B_1} \chi_{B_2}$, $\chi_{B_1 \cup B_2} = \chi_{B_1} + \chi_{B_2}$ if $B_1 \cap B_2 = \emptyset$, $\chi_{B_1 \setminus B_2} = \chi_{B_1} - \chi_{B_2}$ if $B_2 \subset B_1$, and $\chi_{B_1 \triangle B_2} = \chi_{B_1} + \chi_{B_2} \bmod 2$.

σ-algebra

A standard requirement for a measured space $(X, \mathcal{B}, \mu)$ is that $\mathcal{B}$ be a σ-algebra.

Definition. An algebra of sets is called a σ-**algebra** if it is closed under taking countable unions.

Examples of σ-algebras: • $\{\emptyset, X\}$;
• all subsets of X (2^X);
• all finite and countable subsets of X and their complements.

Proposition Given a collection S of subsets of X, there exists a minimal σ-algebra of subsets of X that contains S.

Suppose X is a topological space. The **Borel** σ-algebra $\mathcal{B}(X)$ is the minimal σ-algebra that contains all open subsets of X. Elements of $\mathcal{B}(X)$ are called **Borel sets**. A mapping $F : X \to X$ is measurable relative to $\mathcal{B}(X)$ if and only if the preimage of any open set is Borel. In particular, each continuous map is measurable.

σ-additive measure

Definition. Suppose $\mathcal{B}$ is an algebra of subsets of a set X. A function $\mu : \mathcal{B} \to [0, \infty]$ is an **additive measure** if $\mu(\emptyset) = 0$ and, for any disjoint sets $A_1, A_2, \dots, A_n \in \mathcal{B}$,

$$\mu\Big(\bigcup_{k=1}^{n} A_k\Big) = \sum_{k=1}^{n} \mu(A_k).$$

In the case $\mathcal{B}$ is a σ-algebra, the additive measure μ is σ-**additive** if for any disjoint sets $A_1, A_2, \dots$ from $\mathcal{B}$,

$$\mu\Big(\bigcup_{k=1}^{\infty} A_k\Big) = \sum_{k=1}^{\infty} \mu(A_k).$$

The measure μ is **finite** if $\mu(X) < \infty$. μ is σ-**finite** if $X = \bigcup_{k=1}^{\infty} X_k$, where $\mu(X_k) < \infty$ for all k.

Another standard requirement for a measured space $(X, \mathcal{B}, \mu)$ is that μ be a σ-additive measure and also be finite or σ-finite.

Definition. A normalized invariant **mean** on $\mathbb{Z}$ is a function $\mathfrak{m} : 2^{\mathbb{Z}} \to [0, \infty)$ such that

- $\mathfrak{m}(\emptyset) = 0$, $\mathfrak{m}(\mathbb{Z}) = 1$;
- if $A_1, A_2, \dots, A_k$ are disjoint subsets of $\mathbb{Z}$ then $\mathfrak{m}(A_1 \cup \dots \cup A_k) = \mathfrak{m}(A_1) + \dots + \mathfrak{m}(A_k)$;
- $\mathfrak{m}(n + S) = \mathfrak{m}(S)$ for all $n \in \mathbb{Z}$ and $S \subset \mathbb{Z}$.

The mean $\mathfrak{m}$ is a finite, additive measure on $\mathbb{Z}$. Note that $\mathfrak{m}(\{n\})$ is the same for all $n \in \mathbb{Z}$. Since $\mathfrak{m}(\mathbb{Z}) < \infty$, it follows that $\mathfrak{m}(\{n\}) = 0$. Besides, it follows that $\mathfrak{m}$ is not σ-additive.

Theorem (Banach) There exists a normalized invariant mean on $\mathbb{Z}$.

That is, the group $\mathbb{Z}$ is **amenable**.

Recurrence

$(X, \mathcal{B}, \mu)$: measured space
$T : X \to X$: measure-preserving mapping

Let E be a measurable subset of X. A point $x \in E$ is called **recurrent** if $T^n(x) \in E$ for some $n \geq 1$.

A point $x \in E$ is called **infinitely recurrent** if the orbit $x, T(x), T^2(x), \ldots$ visits E infinitely many times.

Theorem (Poincaré 1890) Suppose μ is a finite measure. Then almost all points of E are infinitely recurrent.

Lemma Suppose μ is a finite measure and $\mu(E) > 0$. Then there exists a recurrent point $x \in E$.

Proof: Let $E_0 = E$, $E_1 = T^{-1}(E)$, $E_2 = T^{-1}(E_1) = T^{-2}(E)$, $\dots$, $E_n = T^{-1}(E_{n-1}) = T^{-n}(E)$, $\dots$

Suppose $E_n \cap E_m \neq \emptyset$ for some n and m, $0 \leq n < m$. Take any point $x \in E_n \cap E_m$ and let $y = T^n(x)$.

Since $T^n(x), T^m(x) \in E$, it follows that $y \in E$ and $T^{m-n}(y) \in E$, hence y is a recurrent point.

Now assume that sets $E_0, E_1, E_2, \dots$ are disjoint. Since T preserves measure, we have $\mu(E_{n+1}) = \mu(E_n)$ for all $n \geq 0$ so that $\mu(E_n) = \mu(E) > 0$ for all n.

Then $\mu(E_0 \cup E_1 \cup E_2 \cup \dots) = \infty$, a contradiction.

Individual ergodic theorem

Let $(X, \mathcal{B}, \mu)$ be a measured space and $T : X \to X$ be a measure-preserving transformation.

Birkhoff's Ergodic Theorem For any function $f \in L_1(X, \mu)$, the limit

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} f(T^k(x)) = f^*(x)$$

exists for almost all $x \in X$. The function f^* is T-invariant, i.e., $f^* \circ T = f^*$ almost everywhere. If μ is finite then $f^* \in L_1(X, \mu)$ and

$$\int_X f^* \, d\mu = \int_X f \, d\mu.$$