MATH 614

Dynamical Systems and Chaos

Lecture 7:
Symbolic dynamics (continued).

Symbolic dynamics

Given a finite set $\mathcal{A}$ (an alphabet), we denote by $\Sigma_{\mathcal{A}}$ the set of all infinite words over $\mathcal{A}$, i.e., infinite sequences $\mathbf{s} = (s_1 s_2 \dots)$, $s_i \in \mathcal{A}$.

For any finite word w over the alphabet $\mathcal{A}$, that is, $w = s_1 s_2 \dots s_n$, $s_i \in \mathcal{A}$, we define a **cylinder** $C(w)$ to be the set of all infinite words $\mathbf{s} \in \Sigma_{\mathcal{A}}$ that begin with w. The topology on $\Sigma_{\mathcal{A}}$ is defined so that open sets are unions of cylinders. Two infinite words are considered close in this topology if they have a long common beginning.

The **shift** transformation $\sigma : \Sigma_{\mathcal{A}} \to \Sigma_{\mathcal{A}}$ is defined by $\sigma(s_0 s_1 s_2 \dots) = (s_1 s_2 \dots)$. This transformation is continuous. The study of the shift and related transformations is called **symbolic dynamics**.

Properties of the shift

- The shift transformation $\sigma : \Sigma_{\mathcal{A}} \to \Sigma_{\mathcal{A}}$ is continuous.

- An infinite word $\mathbf{s} \in \Sigma_{\mathcal{A}}$ is a periodic point of the shift if and only if $\mathbf{s} = www\dots$ for some finite word w.

- An infinite word $\mathbf{s} \in \Sigma_{\mathcal{A}}$ is an eventually periodic point of the shift if and only if $\mathbf{s} = uwww\dots$ for some finite words u and w.

- The shift σ has periodic points of all (prime) periods.

- Periodic points of the shift are dense in $\Sigma_{\mathcal{A}}$.

- The shift transformation $\sigma : \Sigma_{\mathcal{A}} \to \Sigma_{\mathcal{A}}$ admits a dense orbit.

Applications of symbolic dynamics

Suppose $f : X \to X$ is a dynamical system. Given a partition of the set X into disjoint subsets X_α, $\alpha \in \mathcal{A}$ indexed by elements of a finite set $\mathcal{A}$, we can define the **itinerary map** $S : X \to \Sigma_\mathcal{A}$ so that $S(x) = (s_0 s_1 s_2 \dots)$, where $f^n(x) \in X_{s_n}$ for all $n \geq 0$.

In the case f is continuous, the itinerary map is continuous if the sets X_α are **clopen** (i.e., both closed and open).

Indeed, for any finite word $w = s_0 s_1 \dots s_k$ over the alphabet $\mathcal{A}$ the preimage of the cylinder $C(w)$ under the itinerary map is

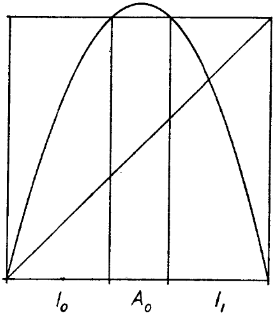
$$S^{-1}(C(w)) = X_{s_0} \cap f^{-1}(X_{s_1}) \cap \dots \cap (f^k)^{-1}(X_{s_k}).$$

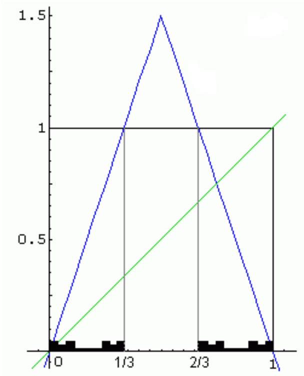
Applications of symbolic dynamics

A more general construction is to take disjoint sets $X_\alpha,\ \alpha \in \mathcal{A}$ that need not cover the entire set X. Then the itinerary map is defined on a subset of X consisting of all points whose orbits stay in the union of the sets X_α.

In the case X is an interval, a partition into clopen sets is not possible. Instead, we choose the sets X_α to be closed intervals with disjoint interiors. Then the itinerary map is not (uniquely) defined on a countable set.

Examples





Any real number x is uniquely represented as $x = k + r$, where $k \in \mathbb{Z}$ and and $0 \le r < 1$. Then k is called the **integer part** of x and r is called the **fractional part** of x. Notation: $k = [x]$, $r = \{x\}$.

Example. $f : [0,1) \to [0,1)$, $f(x) = \{10x\}$.

Consider a partition of the interval $[0,1)$ into 10 subintervals $X_i = [\frac{i}{10}, \frac{i+1}{10})$, $0 \le i \le 9$. That is, $X_0 = [0, 0.1)$, $X_1 = [0.1, 0.2)$,..., $X_9 = [0.9, 1)$.

Given a point $x \in [0,1)$, let $S(x) = (s_0 s_1 s_2 \dots)$ be the itinerary of x relative to that partition. Then $0.s_0 s_1 s_2 \dots$ is the decimal expansion of the real number x.

Subshift

Suppose Σ' is a closed subset of the space $\Sigma_{\mathcal{A}}$ invariant under the shift σ, i.e., $\sigma(\Sigma') \subset \Sigma'$. The restriction of the shift σ to the set Σ' is called a **subshift**.

Examples. • Orbit closure $\overline{O^+_\sigma(\mathbf{s})}$ is always shift-invariant.

• Let $\mathcal{A} = \{0, 1\}$ and Σ' consists of $(00\dots)$, $(11\dots)$, and all sequences of the form $(0\dots011\dots)$. Then Σ' is a closed, shift-invariant set that is not an orbit closure.

• Suppose W is a collection of finite words in the alphabet $\mathcal{A}$. Let Σ' be the set of all $\mathbf{s} \in \Sigma_{\mathcal{A}}$ that do not contain any element of W as a subword. Then Σ' is a closed, shift-invariant set. Any subshift can be defined this way. In the previous example, $W = \{10\}$.

• In the case the set W of "forbidden" words is finite, the subshift is called a **subshift of finite type**.

Random dynamical system

Let f_0 and f_1 be two transformations of a set X. Consider a random dynamical system $F : X \to X$ defined by $F(x) = f_\xi(x)$, where ξ is a random variable taking values 0 and 1.

The symbolic dynamics allows to redefine this dynamical system as a deterministic one. The phase space of the new system is $X \times \Sigma_{\{0,1\}}$ and the transformation is given by

$$\mathcal{F}(x, \mathbf{s}) = \big(f_{s_1(\mathbf{s})}(x), \sigma(\mathbf{s})\big),$$

where $s_1(\mathbf{s})$ is the first entry of the sequence $\mathbf{s}$.

Substitutional dynamical systems

Given a finite set $\mathcal{A}$, let $\mathcal{A}^*$ denote the set of all finite words over the alphabet $\mathcal{A}$ (including the empty word $\varnothing$).

Consider a map $\tau : \mathcal{A} \to \mathcal{A}^* \setminus \{\varnothing\}$, which is referred to as a **substitution rule**. This map can be extended to transformations of $\mathcal{A}^*$ and $\Sigma_{\mathcal{A}}$ according to the rule $\tau(s_1 s_2 \dots) = \tau(s_1)\tau(s_2)\dots$ (substitutional dynamical system).

If two infinite words $\omega, \eta \in \Sigma_{\mathcal{A}}$ have common beginning w, then the words $\tau(\omega)$ and $\tau(\eta)$ have common beginning $\tau(w)$. Note that the length of $\tau(w)$ is not less than the length of w. It follows that the transformation $\tau : \Sigma_{\mathcal{A}} \to \Sigma_{\mathcal{A}}$ is continuous.

Substitutional dynamical systems

Example. $\mathcal{A} = \{a, b, c, d\}$,
$\tau(a) = aca$, $\tau(b) = d$, $\tau(c) = b$, $\tau(d) = c$.

$$\begin{aligned} \tau(a) &= aca, \\ \tau^2(a) &= acabaca, \\ \tau^3(a) &= acabacadacabaca, \\ \dots \end{aligned}$$

Finite words $\tau^n(a)$ "converge" to an infinite word $\omega \in \Sigma_{\mathcal{A}}$ as $n \to \infty$. The infinite word $\omega = acabacadacabaca\dots$ is a unique fixed point of the substitution τ. This fixed point is attracting, namely, $\tau^n(\xi) \to \omega$ for any $\xi = a\dots \in \Sigma_{\mathcal{A}}$.

Let v be a nonempty word over $\{b, c, d\}$. Then $v\omega$ is a periodic point of period 3. For any $\xi = va\dots \in \Sigma_{\mathcal{A}}$ the orbit $\xi, \tau(\xi), \tau^2(\xi), \dots$ is attracted to the cycle $v\omega, \tau(v)\omega, \tau^2(v)\omega$.