MATH 614
Dynamical Systems and Chaos

Lecture 10:
Chaos.
Structural stability.

Definition of chaos

Suppose $f : X \to X$ is a continuous transformation of a metric space (X, d).

Definition. We say that the map f is **chaotic** if

- f has sensitive dependence on initial conditions;
- f is topologically transitive;
- periodic points of f are dense in X.

Theorem For continuous transformations of compact metric spaces, chaoticity is preserved under topological conjugacy.

Separation of orbits

Suppose $f : X \to X$ is a continuous transformation of a metric space (X, d).

Definition. We say that f has **sensitive dependence on initial conditions** if there is $\delta > 0$ such that, for any $x \in X$ and a neighborhood U of x, there exist $y \in U$ and $n \geq 0$ satisfying $d\big(f^n(y), f^n(x)\big) > \delta$.

We say that the map f is **expansive** if there is $\delta > 0$ such that, for any $x, y \in X$, $x \neq y$, there exists $n \geq 0$ satisfying $d\big(f^n(y), f^n(x)\big) > \delta$.

Proposition If X is compact, then changing the metric d to another metric that induces the same topology cannot affect sensitive dependence on i.c. and expansiveness of the map f.

Corollary For continuous transformations of compact metric spaces, sensitive dependence on initial conditions and expansiveness are preserved under topological conjugacy.

Topological transitivity

Suppose $f : X \to X$ is a continuous transformation of a topological space X.

Definition. The map f is **topologically transitive** if for any nonempty open sets $U, V \subset X$ there exists a natural number n such that $f^n(U) \cap V \neq \emptyset$.

$$U \ni x \longmapsto f(x) \longmapsto f^2(x) \longmapsto \cdots \longmapsto f^n(x) \in V$$

Proposition 1 Topological transitivity is preserved under topological conjugacy.

Proposition 2 If the map f has a dense orbit, then it is topologically transitive provided X is Hausdorff and has no isolated points.

Proposition 3 If X is a metrizable compact space, then any topologically transitive transformation of X has a dense orbit.

The shift

Suppose $\mathcal{A}$ is a finite alphabet consisting of at least 2 letters.

Theorem The shift $\sigma : \Sigma_{\mathcal{A}} \to \Sigma_{\mathcal{A}}$ is chaotic.

We already know that periodic points of the shift are dense in $\Sigma_{\mathcal{A}}$. Also, the shift admits a dense orbit and hence is topologically transitive. It remains to check sensitive dependence on initial conditions.

Lemma The shift is expansive.

Idea of the proof: If $\mathbf{s}, \mathbf{t} \in \Sigma_{\mathcal{A}}$ are distinct infinite words, then for some $n \geq 0$ the shifted words $\sigma^n(\mathbf{s})$ and $\sigma^n(\mathbf{t})$ differ in the first letter.

Finally, expansiveness implies sensitive dependence on initial conditions unless the phase space has isolated points.

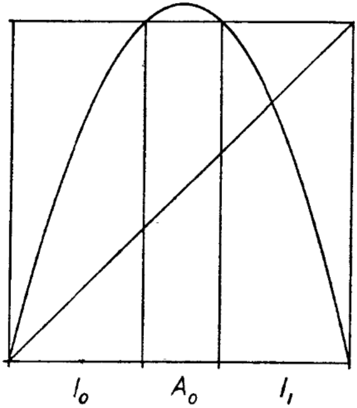
The subshift

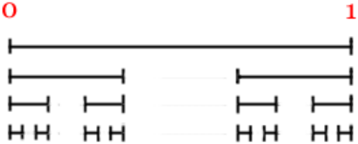
Recall that a subshift is the restriction of the shift to a closed invariant subset.

Clearly, the subshift is expansive as well, but it can have isolated points. Besides, the subshift may or may not admit a dense orbit and periodic points may or may not be dense in the phase space.

All three conditions of chaoticity can be effectively checked for subshifts of finite type.

Unimodal maps





Let $f : \mathbb{R} \to \mathbb{R}$ be a unimodal map, Λ be the set of all points $x \in \mathbb{R}$ such that $O_f^+(x) \subset [0,1]$, and $S : \Lambda \to \Sigma_{\{0,1\}}$ be the itinerary map. We know that S is a continuous semi-conjugacy of the restriction $f|_\Lambda$ with the shift.

Theorem 1 The following conditions are equivalent:
(i) Λ is a Cantor set;
(ii) the itinerary map S is one-to-one;
(iii) the restriction $f|_\Lambda$ is topologically conjugate to the shift;
(iv) the restriction $f|_\Lambda$ has sensitive dependence on initial conditions;
(v) the restriction $f|_\Lambda$ is expansive;
(vi) the restriction $f|_\Lambda$ admits a dense orbit;
(vii) periodic points of f are dense in Λ;
(viii) the restriction $f|_\Lambda$ is chaotic.

Theorem 2 Suppose f is continuously differentiable and $|(f^n)'(x)| > 1$ for some $n \geq 1$ and all $x \in \Lambda$. Then the restriction $f|_\Lambda$ is expansive.

Structural stability

Informally, a dynamical system is **structurally stable** if its structure is preserved under small perturbations. To make this notion formal, one has to specify what it means that the "structure is preserved" and what is considered a "small perturbation".

In the context of topological dynamics, structural stability usually means that the perturbed system is topologically conjugate to the original one.

The description of small perturbations varies for different dynamical systems and so there are various kinds of structural stability.

- Structural stability within a parametric family.

Suppose $f_{\mathbf{p}} : X_{\mathbf{p}} \to X_{\mathbf{p}}$ is a dynamical system depending on a parameter vector $\mathbf{p} \in P$, where $P \subset \mathbb{R}^k$. Given $\mathbf{p}_0 \in P$, we say that $f_{\mathbf{p}_0}$ is **structurally stable within the family** $\{f_{\mathbf{p}}\}$ if there exists $\varepsilon > 0$ such that for any $\mathbf{p} \in P$ satisfying $|\mathbf{p} - \mathbf{p}_0| < \varepsilon$ the system $f_{\mathbf{p}}$ is topologically conjugate to $f_{\mathbf{p}_0}$.

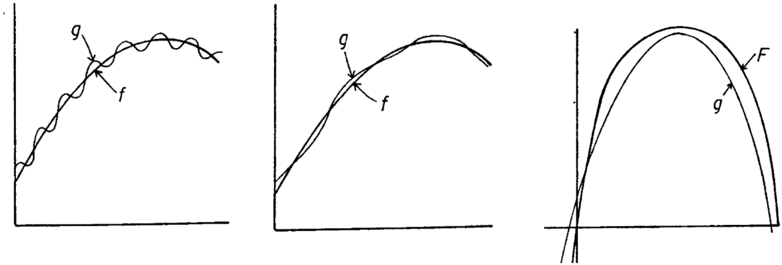
- C^r-structural stability for one-dimensional systems.

Let J be an interval of the real line. For any integer $r \geq 0$, let $C^r(J)$ denote the set of r times continuously differentiable functions $f : J \to \mathbb{R}$. The C^r distance between functions $f, g \in C^r(J)$ is given by

$$d_r(f,g) = \sup_{x \in J}\big(|f(x)-g(x)|, |f'(x)-g'(x)|, \ldots, |f^{(r)}(x)-g^{(r)}(x)|\big).$$

We say that a map $f \in C^r(J)$ is C^r-**structurally stable** if there exists $\varepsilon > 0$ such that whenever $d_r(f,g) < \varepsilon$, it follows that g is topologically conjugate to f.

Small perturbation

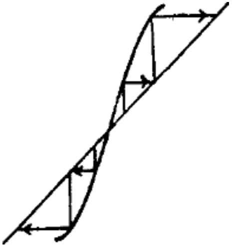


In the first figure, the function g is a C^0-small perturbation of f, but not a C^1-small one. In the second figure, the functions f and g are C^1-close but not C^2-close. In the third figure, f and g are C^2-close.

Examples of structural stability

- Linear map $f_\lambda : \mathbb{R} \to \mathbb{R}$, $f_\lambda(x) = \lambda x$.

The map f_λ is structurally stable within the family $\{f_\lambda\}$ if and only if $\lambda \notin \{-1, 0, 1\}$. Besides, it is C^1-structurally stable for the same values of λ.



Examples of structural stability

- Logistic map $F_\mu : \mathbb{R} \to \mathbb{R}$, $F_\mu(x) = \mu x(1-x)$.

The map F_μ is structurally stable within the family $\{F_\mu\}$ for $\mu > 4$. Besides, it is C^2-structurally stable for $\mu > 4$ (but not C^1-structurally stable).