MATH 614

Dynamical Systems and Chaos

Lecture 11:
Sharkovskii's theorem.

Period set

Suppose J is an interval of the real line and $f : J \to J$ is a continuous map. Let $\mathcal{P}(f)$ be the set of all natural numbers n for which the map f admits a periodic point of prime period n (or, equivalently, a periodic orbit that consists of n points).

Question. Which subsets of $\mathbb{N}$ can occur as $\mathcal{P}(f)$?

Examples. • $f : \mathbb{R} \to \mathbb{R}$, $f(x) = x + 1$.
$\mathcal{P}(f) = \emptyset$.

• $f : \mathbb{R} \to \mathbb{R}$, $f(x) = x$.
$\mathcal{P}(f) = \{1\}$.

• $f : \mathbb{R} \to \mathbb{R}$, $f(x) = -x$.
$\mathcal{P}(f) = \{1, 2\}$.

• $f : \mathbb{R} \to \mathbb{R}$, $f(x) = \mu x(1 - x)$, where $\mu > 4$.
The map f has an invariant set Λ such that the restriction $f|_\Lambda$ is conjugate to the shift on $\Sigma_{\{0,1\}}$. Since the shift admits periodic points of all prime periods, so does f: $\mathcal{P}(f) = \mathbb{N}$.

Sharkovskii's ordering

The **Sharkovskii ordering** is the following strict linear ordering of the natural numbers:

$$\begin{array}{ccccccccccc}
 & 3 & \rhd & 5 & \rhd & 7 & \rhd & 9 & \rhd & \ldots \\
\rhd & 2\cdot 3 & \rhd & 2\cdot 5 & \rhd & 2\cdot 7 & \rhd & 2\cdot 9 & \rhd & \ldots \\
\rhd & 2^2\cdot 3 & \rhd & 2^2\cdot 5 & \rhd & 2^2\cdot 7 & \rhd & 2^2\cdot 9 & \rhd & \ldots \\
\multicolumn{10}{c}{\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots} \\
\multicolumn{10}{c}{\ldots \rhd 2^k \rhd \ldots \rhd 2^3 \rhd 2^2 \rhd 2 \rhd 1.}
\end{array}$$

To be precise, for any integers $k_1, k_2 \geq 0$ and odd natural numbers p_1, p_2 we let $2^{k_1}p_1 \rhd 2^{k_2}p_2$ if and only if one of the following conditions holds:

- $k_1 = k_2$ and $1 < p_1 < p_2$;
- $p_1, p_2 > 1$ and $k_1 < k_2$;
- $p_1 > 1$ and $p_2 = 1$;
- $p_1 = p_2 = 1$ and $k_1 > k_2$.

Sharkovskii's Theorem

Theorem 1 (Sharkovskii) Suppose $f : J \to J$ is a continuous map of an interval $J \subset \mathbb{R}$. If f admits a periodic point of prime period n and $n \rhd m$ for some $m \in \mathbb{N}$, then f admits a periodic point of prime period m as well.

Definition. A subset $E \subset \mathbb{N}$ is called a **tail** of Sharkovskii's ordering if $n \in E$ and $n \rhd m$ implies $m \in E$ for all $m, n \in \mathbb{N}$.

Sharkovskii's Theorem states that the period set $\mathcal{P}(f)$ is such a tail. For any $n \in \mathbb{N}$ the set $E_n = \{n\} \cup \{m \in \mathbb{N} \mid n \rhd m\}$ is a tail. The only tails that cannot be represented this way are $\{2^n \mid n \geq 0\}$ and the empty set.

Theorem 2 For any tail E of Sharkovskii's ordering there exists a continuous map $f : \mathbb{R} \to \mathbb{R}$ such that $\mathcal{P}(f) = E$.

Remark. For maps of an interval $J \subset \mathbb{R}$, Theorem 2 holds with one exception: if J is bounded and closed, then $\mathcal{P}(f) \neq \emptyset$.

Suppose $f : J \to J$ is a continuous map of an interval $J \subset \mathbb{R}$. Given two closed bounded intervals $I_1, I_2 \subset J$, we write $\boxed{I_1 \to I_2}$ if $f(I_1) \supset I_2$ (i.e., if I_1 covers I_2 under the action of f).

Lemma 1 If $I \to I$, then the interval I contains a fixed point of the map f.

Proof: Let $I = [a, b]$. Since $f(I) \supset I$, there exist $a_0, b_0 \in I$ such that $f(a_0) = a$, $f(b_0) = b$. Then a continuous function $g(x) = f(x) - x$ satisfies $g(a_0) = a - a_0 \leq 0$ and $g(b_0) = b - b_0 \geq 0$. By the Intermediate Value Theorem, we have $g(c) = 0$ for some c between a_0 and b_0. Then $c \in I$ and $f(c) = c$.

Lemma 2 If the map f has a periodic orbit, then it has a fixed point.

Proof: Suppose x is a periodic point of f of prime period n. In the case $n = 1$, we are done. Otherwise let $x_1, x_2, \ldots, x_n$ be the list of all points of the orbit $O_f^+(x)$ ordered so that $x_1 < x_2 < \cdots < x_n$. Note that $f(x_i) \neq x_i$ for all i. In particular, $f(x_1) > x_1$ while $f(x_n) < x_n$.

Let j be the largest index satisfying $f(x_j) > x_j$. Then $j < n$, $f(x_j) \geq x_{j+1}$, and $f(x_{j+1}) \leq x_j$. The Intermediate Value Theorem implies that $[x_j, x_{j+1}] \to [x_j, x_{j+1}]$. By Lemma 1, the map f has a fixed point in the interval $[x_j, x_{j+1}]$.

Lemma 3 If $I \to I'$, then there exists a closed interval $I_0 \subset I$ such that f maps I_0 onto I'.

Proof: Let $I' = [a, b]$. Then $A = I \cap f^{-1}(a)$ and $B = I \cap f^{-1}(b)$ are nonempty compact sets. It follows that the distance function $d(x, y) = |y - x|$ attains its minimum on the set $A \times B$ at some point (x_0, y_0). Note that $x_0 \neq y_0$ since $A \cap B = \emptyset$. Let I_0 denote the closed interval with endpoints x_0 and y_0. Then $I_0 \subset I$, the endpoints of I_0 are mapped to a and b, and no interior point of I_0 is mapped to a or b. The Intermediate Value Theorem implies that $f(I_0) = I'$.

Lemma 4 If $I_1 \to I_2 \to \cdots \to I_n \to I_1$, then there exists a fixed point x of f^n such that $x \in I_1$, $f(x) \in I_2$, ..., $f^{n-1}(x) \in I_n$.

Proof: It follows by induction from Lemma 3 that there exist closed intervals $I_1' \subset I_1$, $I_2' \subset I_2$, ..., $I_n' \subset I_n$ such that f maps I_i' onto I_{i+1}' for $1 \leq i \leq n-1$ and also maps I_n' onto I_1. As a consequence, f^n maps I_1' onto I_1. Lemma 1 implies that f^n has a fixed point $x \in I_1'$. By construction, $f^i(x) \in I_i' \subset I_i$ for $0 \leq i \leq n-1$.

Proposition 5 If the map f has a periodic point of prime period 3, then it has periodic points of any prime period.

Proof: Suppose x_1, x_2, x_3 are points forming a periodic orbit of f, ordered so that $x_1 < x_2 < x_3$. We have that either $f(x_1) = x_2$, $f(x_2) = x_3$, $f(x_3) = x_1$, or else $f(x_1) = x_3$, $f(x_2) = x_1$, $f(x_3) = x_2$. In the first case, let $I_1 = [x_2, x_3]$ and $I_2 = [x_1, x_2]$. Otherwise we let $I_1 = [x_1, x_2]$ and $I_2 = [x_2, x_3]$. Then $\circlearrowright I_1 \rightleftarrows I_2$, i.e., $I_1 \to I_2 \to I_1$ and $I_1 \to I_1$.

The map f has a periodic point of prime period 3. By Lemma 2, it also has a fixed point. To find a periodic point of prime period n, where $n = 2$ or $n \geq 4$, we notice that

$$I_2 \to \underbrace{I_1 \to I_1 \to \cdots \to I_1}_{n-1 \text{ times}} \to I_2.$$

By Lemma 4, there exists $x \in I_2$ such that $f^n(x) = x$ and $f^i(x) \in I_1$ for $1 \leq i \leq n-1$. If $x \notin I_1$, we obtain that n is the prime period of x. Otherwise $x = x_2$, which leads to a contradiction.