MATH 614
Dynamical Systems and Chaos

Lecture 13:
Bifurcation theory.

Bifurcation theory

The object of **bifurcation theory** is to study changes that maps undergo as parameters change.

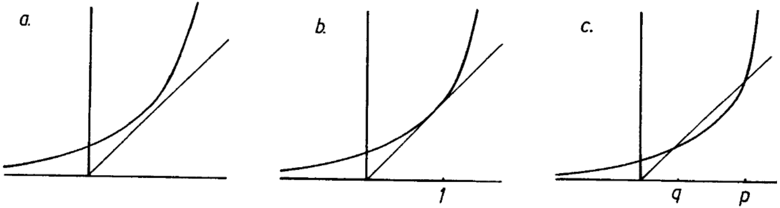
In the context of one-dimensional dynamics, we consider a one-parameter family of maps $f_\lambda : \mathbb{R} \to \mathbb{R}$. We assume that $G(x, \lambda) = f_\lambda(x)$ is smooth a function of two variables.

Informally, the family $\{f_\lambda\}$ has a **bifurcation** at $\lambda = \lambda_0$ if the dynamics of f_λ changes as λ passes λ_0. One way to formalize it is to require that there exist $\varepsilon > 0$ such that for any $\varepsilon_1, \varepsilon_2 \in (0, \varepsilon)$ the maps $f_{\lambda_0 - \varepsilon_1}$ and $f_{\lambda_0 + \varepsilon_2}$ are not topologically conjugate. The simplest case is an isolated bifurcation point λ_0. In this case, the map f_λ is structurally stable for all λ in a punctured neighborhood of λ_0 but not for $\lambda = \lambda_0$.

The condition of topological conjugacy is often relaxed to local topological conjugacy or to similar configuration of periodic orbits.

Saddle-node bifurcation

Exponential map $E_\lambda(x) = \lambda e^x$, $\lambda \approx 1/e$, $x \approx 1$.



For $\lambda > 1/e$, there are no fixed points. At $\lambda = 1/e$, there is a non-hyperbolic fixed point 1. For $0 < \lambda < 1/e$, there are two fixed points, one is repelling and the other one is attracting.

Saddle-node bifurcation

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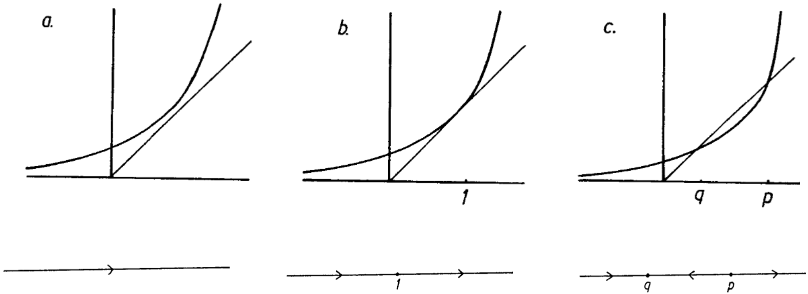
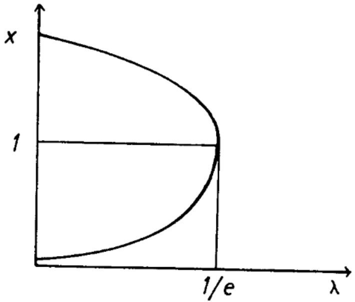


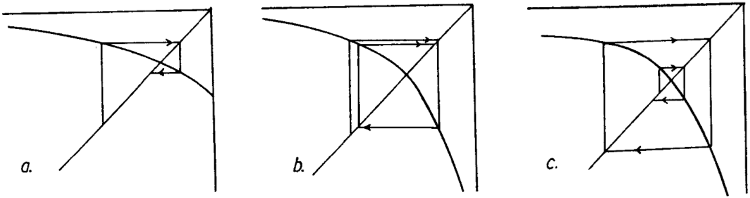
Bifurcation diagram (saddle-node bifurcation)

In the plane with coordinates (λ, x), we plot fixed points of E_λ for each λ:



Period doubling bifurcation

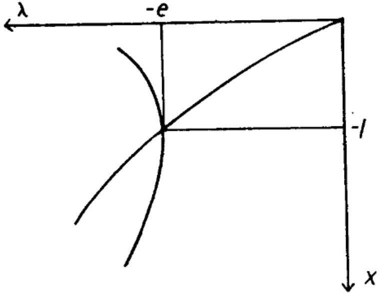
Exponential map $E_\lambda(x) = \lambda e^x$, $\lambda \approx -e$, $x \approx -1$.



For $-e < \lambda < 0$, the fixed point is attracting. At $\lambda = -e$, it is not hyperbolic. For $\lambda < -e$, the fixed point is repelling and there is also an attracting periodic orbit of period 2.

Bifurcation diagram (period doubling bifurcation)

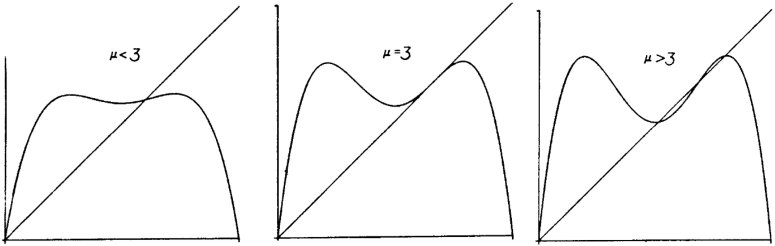
In the plane with coordinates (λ, x), we plot fixed points of E_λ^2 for each λ:



Period doubling: logistic map

Logistic map $F_\mu(x) = \mu x(1-x)$, $\mu \approx 3$, $x \approx 2/3$.

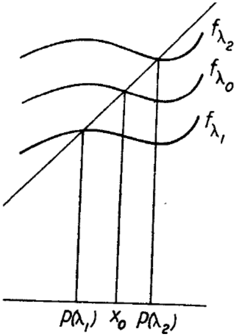
Consider graphs of F_μ^2 for $\mu \approx 3$:

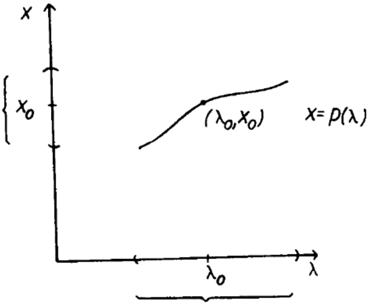


For $\mu < 3$, the fixed point $p_\mu = 1 - \mu^{-1}$ is attracting. At $\mu = 3$, it is not hyperbolic. For $\mu > 3$, the fixed point p_μ is repelling and there is also an attracting periodic orbit of period 2.

No bifurcation: sufficient condition

Theorem 1 Let f_λ be a one-parameter family of functions and suppose that $f_{\lambda_0}(x_0) = x_0$ and $f'_{\lambda_0}(x_0) \neq 1$. Then there are open intervals $I \ni x_0$ and $N \ni \lambda_0$ and a smooth function $p : N \to I$ such that $p(\lambda_0) = x_0$ and $f_\lambda(p(\lambda)) = p(\lambda)$ for all $\lambda \in N$. Moreover, $p(\lambda)$ is the only fixed point of f_λ in I.





No bifurcation: sufficient condition

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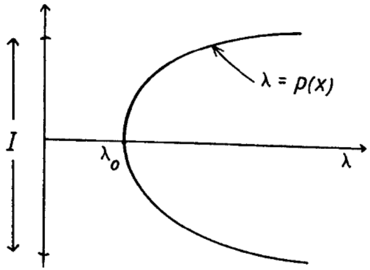
Proof: Consider a function of two variables $G(x, \lambda) = f_\lambda(x) - x$. We have $G(x_0, \lambda_0) = f_{\lambda_0}(x_0) - x_0 = 0$ and $\frac{\partial G}{\partial x}(x_0, \lambda_0) = f'_{\lambda_0}(x_0) - 1 \neq 0$. By the Implicit Function Theorem, there are open intervals $I \ni x_0$ and $N \ni \lambda_0$ and a smooth function $p : N \to I$ such that

$$G(x, \lambda) = 0 \iff x = p(\lambda) \text{ for all } (x, \lambda) \in I \times N.$$

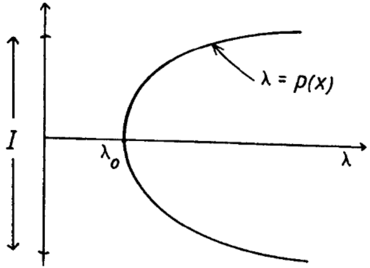
Saddle-node bifurcation: sufficient condition

Theorem 2 Let f_λ be a one-parameter family of functions and suppose that $f_{\lambda_0}(x_0) = x_0$, $f'_{\lambda_0}(x_0) = 1$, $f''_{\lambda_0}(x_0) \neq 0$, and $\frac{\partial f_\lambda}{\partial \lambda}\Big|_{\lambda=\lambda_0}(x_0) \neq 0$. Then there are open intervals $I \ni x_0$ and $N \ni \lambda_0$ and a smooth function $p : I \to N$ such that $p(x_0) = \lambda_0$ and $f_{p(x)}(x) = x$ for all $x \in I$. Moreover, $p'(x_0) = 0$ and $p''(x_0) \neq 0$.



Period doubling bifurcation: sufficient condition

Theorem 3 Let f_λ be a one-parameter family of functions and suppose that $f_\lambda(x_0) = x_0$ for all λ, $f'_{\lambda_0}(x_0) = -1$, and $\left.\frac{\partial (f_\lambda^2)'}{\partial \lambda}\right|_{\lambda=\lambda_0}(x_0) \neq 0$. Then there are open intervals $I \ni x_0$ and $N \ni \lambda_0$ and a smooth function $p : I \to N$ such that $p(x_0) = \lambda_0$ and $f^2_{p(x)}(x) = x$ for all $x \in I$ but $f_{p(x)}(x) \neq x$ for $x \in I \setminus \{x_0\}$.



More examples

- Quadratic maps: $Q_c(x) = x^2 + c$.

The family undergoes a saddle-node bifurcation at $c = 1/4$ and a period doubling bifurcation at $c = -3/4$. It undergoes a lot of other bifurcations as well.

- Hyperbolic sine family: $H_\lambda(x) = \lambda \sinh x$.

A map H_λ is not structurally stable within the family for $\lambda = -1$, 0, and 1. At $\lambda = -1$, we have a period doubling bifurcation. At $\lambda = 1$, the family transitions from one to three fixed points. At $\lambda = 0$, the bifurcation does not change the configuration of periodic points.

- Linear maps: $f_\lambda(x) = \lambda^2 x$.

A map f_λ is not structurally stable within the family for $\lambda = -1$, 0, and 1. At $\lambda = -1$ and 1, the family transitions from a repelling fixed point to an attracting one (or vice versa). At $\lambda = 0$, there is no bifurcation.

Period-doubling route to chaos

The logistic map F_μ has the period doubling bifurcation when the parameter μ passes 3. As μ increases beyond 3, the map undergoes repeated period doublings, namely, the period doubling bifurcation for F_μ^2, then for F_μ^4, then for F_μ^8, and so on.