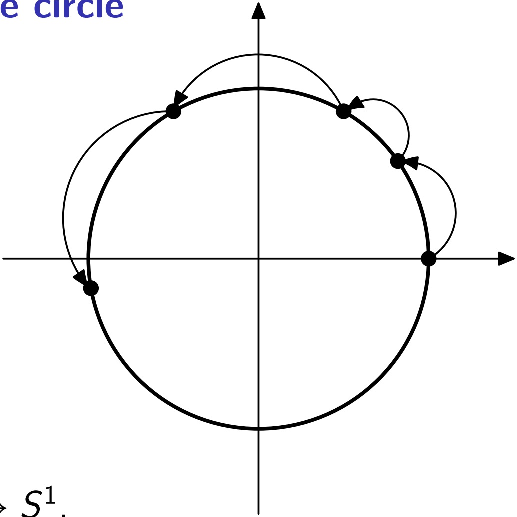
MATH 614

Dynamical Systems and Chaos

Lecture 16:
Rotation number.
The standard family.

Maps of the circle



$T : S^1 \to S^1$,
T an orientation-preserving homeomorphism.

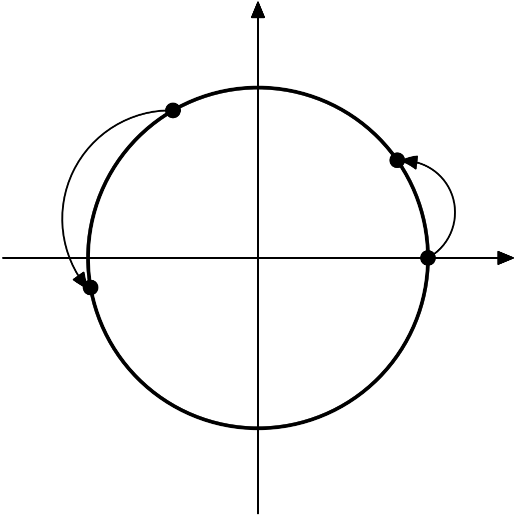
Suppose $T : S^1 \to S^1$ is an orientation-preserving homeomorphism.

Is T topologically conjugate to a rotation R_ω?

Assume this is so, then how can we find ω?

For any $x \in S^1$ let $\omega(T, x)$ denote the length of the shortest arc that goes from x to $T(x)$ in the counterclockwise direction.

If T is a rotation then $\omega(T, x)$ is a constant.



Rotation number

Consider the average

$$A_n(T,x) = \frac{1}{n}\sum_{k=0}^{n-1}\omega(T, T^k(x)).$$

Theorem The limit

$$\lim_{n\to\infty} A_n(T,x)$$

exists for any $x \in S^1$ and does not depend on x.

The **rotation number** of T is

$$\rho(T) = \frac{1}{2\pi}\lim_{n\to\infty} A_n(T,x).$$

Proposition If $\rho(T) = 0$, then T has a fixed point.

Proof: Suppose T has no fixed points. Then $0 < \omega(T, x) < 2\pi$ for any $x \in S^1$. Since $\omega(T, x)$ is a continuous function of x, there exists $\varepsilon > 0$ such that $\varepsilon \leq \omega(T, x) \leq 2\pi - \varepsilon$ for any $x \in S^1$. Then $\varepsilon \leq A_n(T, x) \leq 2\pi - \varepsilon$ for all $x \in S^1$ and $n = 1, 2, \ldots$ It follows that

$$\frac{\varepsilon}{2\pi} \leq \rho(T) \leq 1 - \frac{\varepsilon}{2\pi}.$$

Properties of the rotation number

- For any T, $0 \leq \rho(T) < 1$.

- $\rho(R_\omega) = \omega/(2\pi) \pmod 1$, where R_ω is the rotation by ω.

- If g is an orientation-preserving homeomorphism of S^1, then $\rho(g^{-1}Tg) = \rho(T)$.

- If g is an orientation-reversing homeomorphism of S^1, then $\rho(g^{-1}Tg) = -\rho(T) \pmod 1$.

- If T_1 and T_2 are topologically conjugate, then $\rho(T_1) = \pm\rho(T_2) \pmod 1$.

Properties of the rotation number

- Rotations R_{ω_1} and R_{ω_1} are topologically conjugate if and only if $\omega_1 = \pm\omega_2 \pmod{2\pi}$.

- $\rho(T^n) = n\rho(T) \pmod{1}$.

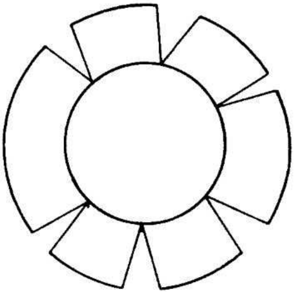
- $\rho(T) = 0$ if and only if T has a fixed point.

- $\rho(T)$ is rational if and only if T has a periodic point.

- If T has a periodic point of prime period n, then $\rho(T) = k/n$, a reduced fraction.

Theorem (Denjoy) If T is C^2 smooth and the rotation number $\rho(T)$ is irrational, then T is topologically conjugate to a rotation of the circle.

Example (Denjoy). There exists C^1 smooth diffeomorphism T of S^1 such that $\rho(T)$ is irrational but T is not minimal.



Proposition Suppose $f : S^1 \to S^1$ is an orientation-preserving homeomorphism. Let $\varepsilon > 0$. Then there exists $\delta > 0$ such that for any homeomorphism $g : S^1 \to S^1$ with

$$\sup_{x \in S^1} \mathrm{dist}(f(x), g(x)) < \delta$$

we have $|\rho(f) - \rho(g)| < \varepsilon \pmod 1$.

Corollary Suppose f_λ is a one-parameter family of orientation-preserving homeomorphisms of S^1. If f_λ depends continuously on λ then $\rho(f_\lambda)$ is a continuous $(\bmod 1)$ function of λ.

The standard family

The **standard** (or **canonical**) family of maps

$$f_{\omega,\varepsilon} : S^1 \to S^1, \quad \omega \in \mathbb{R},\ \varepsilon \geq 0.$$

In the angular coordinate α:

$$f_{\omega,\varepsilon}(\alpha) = \alpha + \omega + \varepsilon \sin \alpha.$$

If $\varepsilon = 0$ then $f_{\omega,\varepsilon} = R_\omega$ is a rotation.
For $0 \leq \varepsilon < 1$, $f_{\omega,\varepsilon}$ is a diffeomorphism.
If $\varepsilon = 1$ then $f_{\omega,\varepsilon}$ is only a homeomorphism.
If $\varepsilon > 1$ then $f_{\omega,\varepsilon}$ is not one-to-one.

The rotation number $\rho(f_{\omega,\varepsilon})$:

- depends continuously $(\mathrm{mod}\, 1)$ on ω and ε;
- is a 2π-periodic function of ω for any ε;
- $f_{0,\varepsilon}$ has rotation number 0;
- $\rho(f_{\omega,\varepsilon})$ is a non-decreasing function of

$\omega \in (0, 2\pi)$ for any fixed ε;

- $\lim_{\omega \to 2\pi} \rho(f_{\omega,\varepsilon}) = 1$.

Hence the map $r_\varepsilon : [0, 1) \to [0, 1)$ given by $x \mapsto \rho(f_{2\pi x,\varepsilon})$ is continuous, non-decreasing, and onto.

r_0 is the identity.

Proposition Suppose $\rho(f_{\omega_0,\varepsilon})$ is rational. If $\varepsilon > 0$ then

$$\rho(f_{\omega,\varepsilon}) = \rho(f_{\omega_0,\varepsilon})$$

for all $\omega > \omega_0$ close enough to ω_0 or for all $\omega < \omega_0$ close enough to ω_0 (or both).

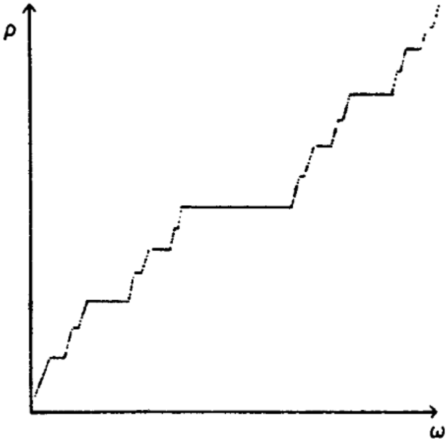
Theorem For any irrational number $0 < \rho_0 < 1$ and any $0 < \varepsilon < 1$, there is exactly one $\omega \in (0, 2\pi)$ such that $\rho(f_{\omega,\varepsilon}) = \rho_0$.

Let $0 < \varepsilon < 1$ and $0 \leq \rho_0 < 1$. Then $r_\varepsilon^{-1}(\rho_0)$ is a point if ρ_0 is irrational and $r_\varepsilon^{-1}(\rho_0)$ is a nontrivial interval if ρ_0 is rational.

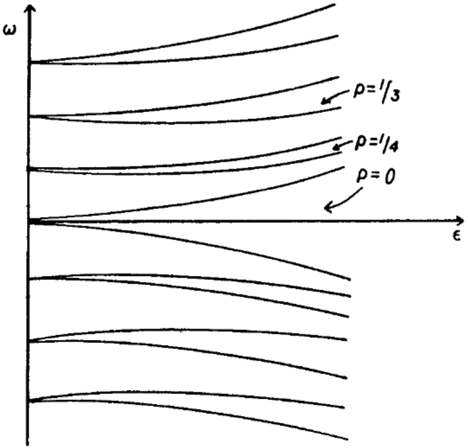
r_ε is a **Cantor function**, which means that on the complement of a Cantor set, $r_\varepsilon' = 0$.

The graph of a Cantor function is called the "**devil's staircase**".

Cantor function



The bifurcation diagram for the standard family



The bifurcation diagram for the standard family

We plot the regions in the (ε, ω)-plane where $\rho(f_{\omega,\varepsilon})$ is a fixed rational number. Each region is a "tongue" that flares from a point $\varepsilon = 0$, $\omega = m/n$, $m, n \in \mathbb{Z}$. None of these tongues can overlap when $\varepsilon < 1$.

Consider the tongue corresponding to $\rho = 0$.
It describes fixed points of the standard maps.
This tongue is the angle $|\omega| \leq \varepsilon$.

What happens when we fix ε and vary ω?

If $\omega = -\varepsilon$ then $f_{\omega,\varepsilon}(\alpha) = \alpha + \omega + \varepsilon \sin \alpha$ has a unique fixed point $\pi/2$. As we increase ω, it splits into two fixed points, one in $(-\pi/2, \pi/2)$, the other in $(\pi/2, 3\pi/2)$. They run around the circle in opposite directions. Finally, at $\omega = \varepsilon$ the two points coalesce into a single fixed point $-\pi/2$.

The unique fixed points for $\omega = \pm\varepsilon$ are neutral. As for two fixed points for $|\omega| < \varepsilon$, one is attracting while the other is repelling (which one?).

So the family $f_{\omega,\varepsilon}$ (ε fixed) enjoys a saddle-node bifurcation two times. Notice that these are not pure saddle-node bifurcations since the bifurcation points are not isolated (they are "half-isolated").