MATH 614

Dynamical Systems and Chaos

Lecture 18: Dynamics of linear maps (continued). Toral endomorphisms.

Linear transformations

Any linear mapping $L : \mathbb{R}^n \to \mathbb{R}^n$ is represented as multiplication of an n-dimensional column vector by a $n \times n$ matrix: $L(\mathbf{x}) = A\mathbf{x}$, where $A = (a_{ij})_{1 \le i,j \le n}$.

Dynamics of linear transformations corresponding to particular matrices is determined by eigenvalues and the Jordan canonical form.

Stable and unstable subspaces

Proposition 1 Suppose that all eigenvalues of a linear map $L : \mathbb{R}^n \to \mathbb{R}^n$ are less than 1 in absolute value. Then $L^n(\mathbf{x}) \to \mathbf{0}$ as $n \to \infty$ for all $\mathbf{x} \in \mathbb{R}^n$.

Proposition 2 Suppose that all eigenvalues of a linear map $L : \mathbb{R}^n \to \mathbb{R}^n$ are greater than 1 in absolute value. Then $L^{-n}(\mathbf{x}) \to \mathbf{0}$ as $n \to \infty$ for all $\mathbf{x} \in \mathbb{R}^n$.

Given a linear map $L : \mathbb{R}^n \to \mathbb{R}^n$, let W^s denote the set of all vectors $\mathbf{x} \in \mathbb{R}^n$ such that $L^n(\mathbf{x}) \to \mathbf{0}$ as $n \to \infty$. In the case L is invertible, let W^u denote the set of all vectors $\mathbf{x} \in \mathbb{R}^n$ such that $L^{-n}(\mathbf{x}) \to \mathbf{0}$ as $n \to \infty$.

Proposition 3 W^s and W^u are vector subspaces of $\mathbb{R}^n$ that are transversal: $W^s \cap W^u = \{\mathbf{0}\}$.

Definition. W^s is called the **stable subspace** of the linear map L. W^u is called the **unstable subspace** of L.

Hyperbolic linear maps

Definition. A linear map L is called **hyperbolic** if it is invertible and all eigenvalues of L are different from 1 in absolute value.

Proposition Suppose $L : \mathbb{R}^n \to \mathbb{R}^n$ is a hyperbolic linear map. Then

- $W^s \oplus W^u = \mathbb{R}^n$;
- if $\mathbf{x} \notin W^s \cup W^u$, then $L^n(\mathbf{x}) \to \infty$ as $n \to \pm\infty$.

Stable and unstable sets

Let $f : X \to X$ be a continuous map of a metric space (X, d).

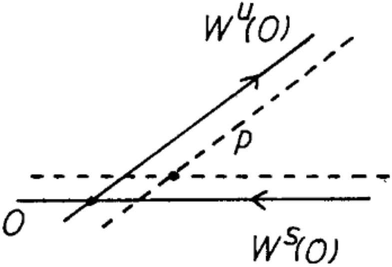
Definition. Two points $x, y \in X$ are **forward asymptotic** with respect to f if $d\big(f^n(x), f^n(y)\big) \to 0$ as $n \to \infty$. The **stable set** of a point $x \in X$, denoted $W^s(x)$, is the set of all points forward asymptotic to x.

Being forward asymptotic is an equivalence relation on X. The stable sets are equivalence classes of this relation. In particular, these sets form a partition of X.

In the case f is a homeomorphism, we say that two points $x, y \in X$ are **backward asymptotic** with respect to f if $d\big(f^{-n}(x), f^{-n}(y)\big) \to 0$ as $n \to \infty$. The **unstable set** of a point $x \in X$, denoted $W^u(x)$, is the set of all points backward asymptotic to x. The unstable set $W^u(x)$ coincides with the stable set of x relative to the inverse map f^{-1}.

Example

- Linear map $L : \mathbb{R}^n \to \mathbb{R}^n$.



The stable and unstable sets of the origin, $W^s(\mathbf{0})$ and $W^u(\mathbf{0})$, are transversal subspaces of the vector space $\mathbb{R}^n$. For any point $\mathbf{p} \in \mathbb{R}^n$, the stable and unstable sets are obtained from $W^s(\mathbf{0})$ and $W^u(\mathbf{0})$ by a translation: $W^s(\mathbf{p}) = \mathbf{p} + W^s(\mathbf{0})$, $W^u(\mathbf{p}) = \mathbf{p} + W^u(\mathbf{0})$.

Real projective plane

Real projective plane $\mathbb{RP}^2$ is obtained from the Euclidean plane by adding points "at infinity".

Formally, elements of $\mathbb{RP}^2$ are one-dimensional subspaces of $\mathbb{R}^3$ (straight lines through the origin). The angle between lines serves as a distance function. Topologically, $\mathbb{RP}^2$ is a closed non-orientable surface.

Lines in the real projective plane correspond to 2-dimensional subspaces of $\mathbb{R}^3$. They are simple closed curves.

Points in the projective plane are given by their **homogeneous coordinates** $[x : y : z]$. The Euclidean plane $\mathbb{R}^2$ is embedded into $\mathbb{RP}^2$ via the map $(x, y) \mapsto [x : y : 1]$.

Projective transformations

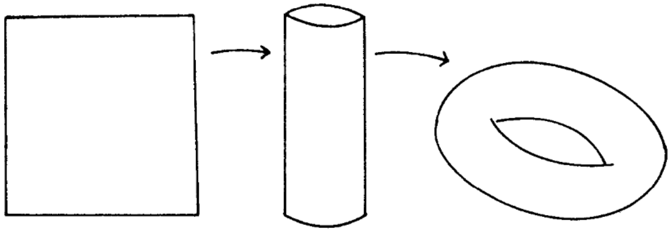
A **projective transformation** of $\mathbb{RP}^2$ is a self-map that takes lines to lines. For any projective transformation P there exists an invertible 3×3 matrix $A = (a_{ij})$ such that $[x' : y' : z'] = P([x : y : z])$ if and only if

$$\begin{pmatrix} x' \\ y' \\ z' \end{pmatrix} = r \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

for some scalar $r \neq 0$. The matrix A is unique up to scaling. Dynamics of P is determined by spectral properties of A.

Torus

The two-dimensional **torus** is a closed surface obtained by gluing together opposite sides of a square by translation.



Torus

The **two-dimensional torus** is a closed surface obtained by gluing together opposite sides of a square by translation.

Alternatively, the torus is defined as $\mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$, the quotient of the plane $\mathbb{R}^2$ by the integer lattice $\mathbb{Z}^2$. To be precise, we introduce a relation on $\mathbb{R}^2$: $\mathbf{x} \sim \mathbf{y}$ if $\mathbf{y} - \mathbf{x} \in \mathbb{Z}^2$. This is an equivalence relation and $\mathbb{T}^2$ is the set of equivalence classes. The plane $\mathbb{R}^2$ induces a distance function, a topology, and a smooth structure on the torus $\mathbb{T}^2$. Also, the addition is well defined on $\mathbb{T}^2$. We denote the equivalence class of a point $(x, y) \in \mathbb{R}^2$ by $[x, y]$.

Topologically, the torus $\mathbb{T}^2$ is the Cartesian product of two circles: $\mathbb{T}^2 = S^1 \times S^1$.

Similarly, the n-**dimensional torus** is defined as $\mathbb{T}^n = \mathbb{R}^n/\mathbb{Z}^n$. Topologically, it is the Cartesian product of n circles: $\mathbb{T}^n = S^1 \times \cdots \times S^1$.

Transformations of the torus

Let $\pi : \mathbb{R}^n \to \mathbb{T}^n$ be the natural projection, $\pi(x_1, \dots, x_n) = [x_1, \dots, x_n]$. Suppose $F : \mathbb{R}^n \to \mathbb{R}^n$ is a transformation such that $\mathbf{x} \sim \mathbf{y} \implies F(\mathbf{x}) \sim F(\mathbf{y})$ for all $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$. Then it gives rise to a unique transformation $f : \mathbb{T}^n \to \mathbb{T}^n$ satisfying $f \circ \pi = \pi \circ F$:

$$
\begin{array}{ccc}
\mathbb{R}^n & \xrightarrow{F} & \mathbb{R}^n \\
\pi \downarrow & & \downarrow \pi \\
\mathbb{T}^n & \xrightarrow{f} & \mathbb{T}^n
\end{array}
$$

The map f is continuous (resp., smooth) if so is F.

Examples. • Translation (or rotation).
$F(\mathbf{x}) = \mathbf{x} + \mathbf{x}_0$, where $\mathbf{x}_0 \in \mathbb{R}^n$ is a constant vector.

• Toral endomorphism.
$F(\mathbf{x}) = A\mathbf{x}$, where A is an $n \times n$ matrix with integer entries.

Translations of the torus

For any vector $\mathbf{v} \in \mathbb{R}^n$ and a point of the n-dimensional torus $\mathbf{x} \in \mathbb{T}^n$, the sum $\mathbf{x} + \mathbf{v}$ is a well-defined element of $\mathbb{T}^n$.

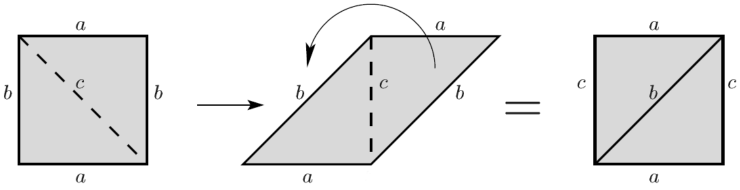
Given $\mathbf{v} \in \mathbb{R}^n$, let $T_{\mathbf{v}}(\mathbf{x}) = \mathbf{x} + \mathbf{v}$ be the **translation** of the torus $\mathbb{T}^n$.

Theorem 1 Let $\mathbf{v} = (v_1, v_2, \dots, v_n)$. The linear flow $T_{t\mathbf{v}}$, $t \in \mathbb{R}$ is minimal (all orbits are dense) if and only if the real numbers $v_1, v_2, \dots, v_n$ are linearly independent over $\mathbb{Q}$. That is, if $r_1 v_1 + \cdots + r_n v_n = 0$ implies $r_1 = \cdots = r_n = 0$ for all $r_1, \dots, r_n \in \mathbb{Q}$.

Theorem 2 Let $\mathbf{v} = (v_1, v_2, \dots, v_n)$. The translation $T_{\mathbf{v}}$ is minimal (all orbits are dense) if and only if the real numbers $1, v_1, v_2, \dots, v_n$ are linearly independent over $\mathbb{Q}$.

Toral automorphisms

$$\textit{Example.} \quad F(\mathbf{x}) = A\mathbf{x}, \quad \text{where} \quad A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}.$$



Hyperbolic toral automorphisms

Suppose A is an $n \times n$ matrix with integer entries. Let L_A denote a toral endomorphism induced by the linear map $L(\mathbf{x}) = A\mathbf{x}$, $\mathbf{x} \in \mathbb{R}^n$. The map L_A is a **toral automorphism** if it is invertible.

Proposition The following conditions are equivalent:

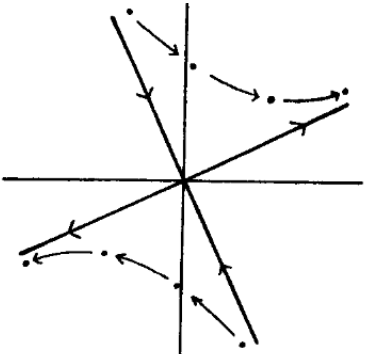
- L_A is a toral automorphism,
- A is invertible and A^{-1} has integer entries,
- $\det A = \pm 1$.

Definition. A toral automorphism L_A is **hyperbolic** if the matrix A has no eigenvalues of absolute value 1.

Theorem Every hyperbolic toral automorphism is chaotic.

Cat map

Example. L_A, where $A = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$.



Stable and unstable subspaces project to dense curves on the torus.