MATH 614

Dynamical Systems and Chaos

Lecture 23: Attractors. Hyperbolic periodic points.

Attractors

Suppose $F : D \to D$ is a topological dynamical system on a metric space D.

Definition. A compact set $N \subset D$ is called a **trapping region** for F if $F(N) \subset \mathrm{int}(N)$.

If N is a trapping region, then $N, F(N), F^2(N), \dots$ are nested compact sets and their intersection Λ is an invariant set: $F(\Lambda) \subset \Lambda$.

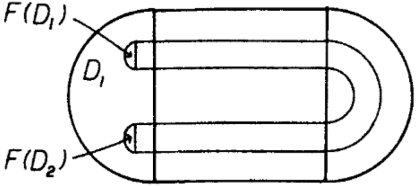
Definition. A set $\Lambda \subset D$ is called an **attractor** for F if there exists a neighborhood N of Λ such that the closure $\overline{N}$ is a trapping region for F and $\Lambda = N \cap F(N) \cap F^2(N) \cap \dots$

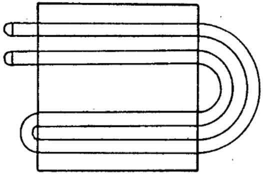
The attractor Λ is **transitive** if the restriction of F to Λ is a transitive map.

Examples of attractors

• Any attracting fixed point or an attracting periodic orbit is a transitive attractor.

• The solenoid is a transitive attractor.





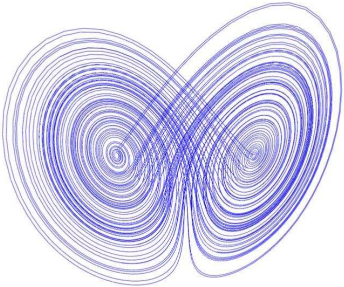
• The horseshoe map has an attractor that is not transitive.

Strange attractors

- The Lorenz attractor.

The Lorenz equations: $$\begin{cases} \dot{x} = \sigma(y - x), \\ \dot{y} = x(\rho - z) - y, \\ \dot{z} = xy - \beta z, \end{cases}$$

where σ, ρ, β are parameters. In the case $\sigma = 10$, $\rho = 28$, $\beta = 8/3$, the system has a "strange" attractor.



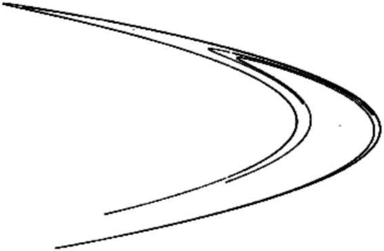
Strange attractors

- The Hénon attractor.

The Hénon map is a simplified version of the first-return map for the Lorenz system:

$$F\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 - ax^2 + y \\ bx \end{pmatrix},$$

where a, b are parameters. In the case $a = 1.4$, $b = 0.3$, the system has a strange attractor.



Hyperbolic periodic points

Suppose $F : \mathbb{R}^n \to \mathbb{R}^n$ is a differentiable map.

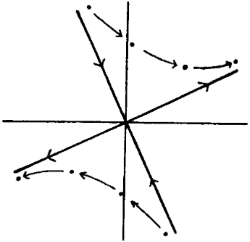
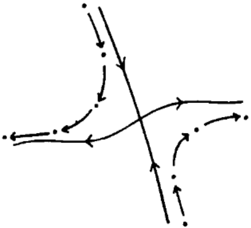
Definition. A fixed point p of the map F is **hyperbolic** if the Jacobian matrix $DF(p)$ has no eigenvalues of absolute value 1 or 0. A periodic point p of period n of the map F is **hyperbolic** if p is a hyperbolic fixed point of the map F^n.

Notice that $DF^n(p) = DF(F^{n-1}(p)) \dots DF(F(p))\, DF(p)$.
It follows that $DF^n(p), DF^n(F(p)), \dots, DF^n(F^{n-1}(p))$ are similar matrices. In particular, they have the same eigenvalues.

Definition. The hyperbolic periodic point p of period n is a **sink** if every eigenvalue λ of $DF^n(p)$ satisfies $0 < |\lambda| < 1$, a **source** if every eigenvalue λ satisfies $|\lambda| > 1$, and a **saddle point** otherwise.

Saddle point

The following figures show the phase portrait of a linear and a nonlinear two-dimensional maps near a saddle point.

Stable and unstable manifolds

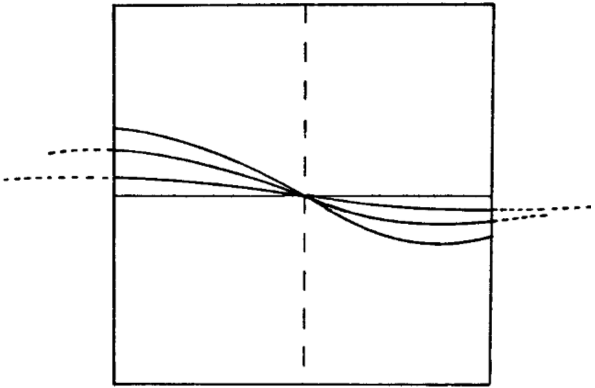
Let $F : \mathbb{R}^2 \to \mathbb{R}^2$ be a diffeomorphism and suppose p is a saddle point of F of period m.

Theorem There exists a smooth curve $\gamma : (-\varepsilon, \varepsilon) \to \mathbb{R}^2$ such that

(i) $\gamma(0) = p$;
(ii) $\gamma'(0)$ is an unstable eigenvector of $DF^m(p)$;
(iii) $F^{-1}(\gamma) \subset \gamma$;
(iv) $\|F^{-n}(\gamma(t)) - F^{-n}(p)\| \to 0$ as $n \to \infty$.
(v) $\|F^{-n}(x) - F^{-n}(p)\| < \varepsilon$ for all $n \geq 0$, then $x = \gamma(t)$ for some t.

The curve γ is called the **local unstable manifold** of F at p. The **local stable manifold** of F at p is defined as the local unstable manifold of F^{-1} at p.

Stable and unstable manifolds

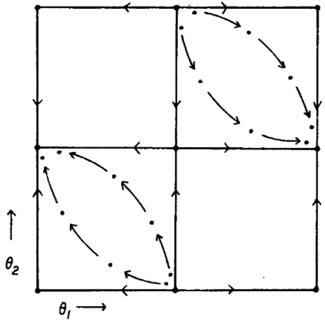


Example

In angular coordinates (θ_1, θ_2) on the torus $\mathbb{T}^2 = (\mathbb{R}/2\pi\mathbb{Z})^2$,

$$F\begin{pmatrix}\theta_1\\ \theta_2\end{pmatrix} = \begin{pmatrix}\theta_1 - \varepsilon \sin\theta_1\\ \theta_2 + \varepsilon\sin\theta_2\end{pmatrix}.$$

There are 4 fixed points: one source, one sink, and two saddles.



Example

In angular coordinates (θ_1, θ_2) on the torus $\mathbb{T}^2 = (\mathbb{R}/2\pi\mathbb{Z})^2$,

$$F\begin{pmatrix}\theta_1\\ \theta_2\end{pmatrix} = \begin{pmatrix}\theta_1 + \varepsilon\sin\theta_1\\ \theta_2 + \varepsilon\sin\theta_2\cos\theta_1\end{pmatrix}.$$

There are 4 fixed points: one source, one sink, and two saddles.