MATH 614

Dynamical Systems and Chaos

Lecture 24:
Bifurcation theory in higher dimensions. The Hopf bifurcation.

Bifurcation theory

The object of **bifurcation theory** is to study changes that maps undergo as parameters change.

In the context of higher-dimensional dynamics, we consider a one-parameter family of maps $F_\lambda : \mathbb{R}^n \to \mathbb{R}^n$. We assume that $G(\mathbf{x}, \lambda) = F_\lambda(\mathbf{x})$ is a smooth function of $n+1$ variables.

We say that the family $\{F_\lambda\}$ undergoes a **bifurcation** at $\lambda = \lambda_0$ if the configuration of periodic points (or, more generally, invariant sets) of F_λ changes as λ passes λ_0.

The simplest examples of bifurcations in higher dimensions occur when we consider a family of the form

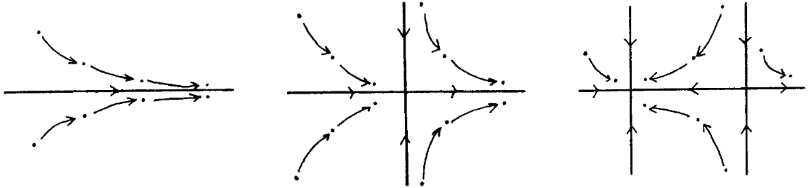
$$F_\lambda(x_1, x_2, \dots, x_n) = \big(f_{1,\lambda}(x_1), f_{2,\lambda}(x_2), \dots, f_{n,\lambda}(x_n)\big),$$

that is, the Cartesian product of n one-dimensional families, when one of those families undergoes a bifurcation at $\lambda = \lambda_0$.

Saddle-node bifurcation

Example. $F_\lambda(x,y) = \big(f_\lambda(x), g_\lambda(y)\big)$, where $f_\lambda(x) = e^x - \lambda$, $g_\lambda(y) = \frac{1}{2}\lambda \arctan y$.

The family f_λ undergoes a saddle-node bifurcation at $\lambda = 1$.

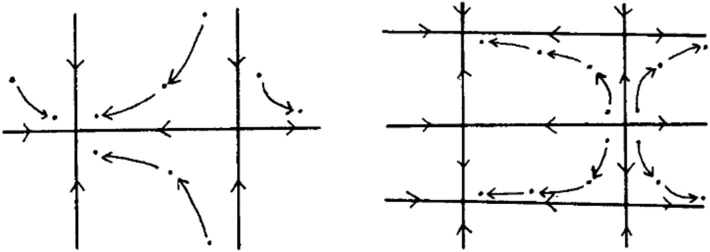


The figures show phase portraits of maps F_λ near $\lambda = 1$.

Period doubling bifurcation

Example. $F_\lambda(x,y) = \big(f_\lambda(x), h_\lambda(y)\big)$, where $f_\lambda(x) = e^x - \lambda$, $h_\lambda(y) = -\frac{1}{2}\lambda \arctan y$.

The family h_λ undergoes a period doubling bifurcation at $\lambda = 2$.



The figures show phase portraits of maps F_λ^2 near $\lambda = 2$.

Hyperbolic fixed points

Let $F : \mathbb{R}^n \to \mathbb{R}^n$ be a smooth map. We denote by $DF(\mathbf{x})$ the **Jacobian matrix** of the map F at $\mathbf{x}$. It is an $n \times n$ matrix whose entries are partial derivatives of F at $\mathbf{x}$.

Definition. A fixed point $\mathbf{x}_0$ of the map F is called **hyperbolic** if the Jacobian matrix $DF(\mathbf{x}_0)$ is hyperbolic, that is, if it has no eigenvalues of absolute value 1 or 0.

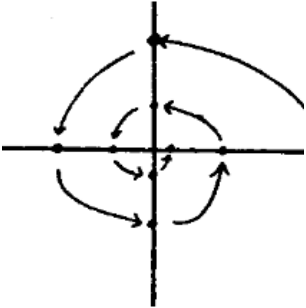
Theorem (Grobman-Hartman) If the fixed point $\mathbf{x}_0$ of the map F is hyperbolic, then in a neighborhood of $\mathbf{x}_0$ the map F is topologically conjugate to a linear map $G(\mathbf{x}) = A\mathbf{x}$, where $A = DF(\mathbf{x}_0)$.

As a consequence, a family of maps F_λ fixing a point $\mathbf{x}_0$ can undergo a bifurcation at $\lambda = \lambda_0$ in a neighborhood of $\mathbf{x}_0$ only if the fixed point $\mathbf{x}_0$ is not hyperbolic for F_{λ_0}.

Example

In polar coordinates (r, θ),

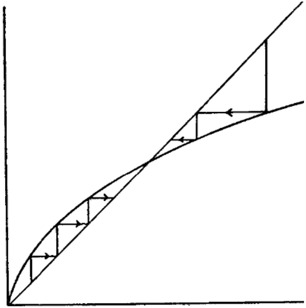
$F_\lambda(r, \theta) = (r_1, \theta_1)$, where $r_1 = \lambda r$, $\theta_1 = \theta + \alpha$.



The maps F_λ, $\lambda > 0$ are linear, with complex conjugate eigenvalues $\lambda e^{\pm i\alpha}$. The origin is a fixed point. It changes from an attracting to a repelling one as λ passes 1.

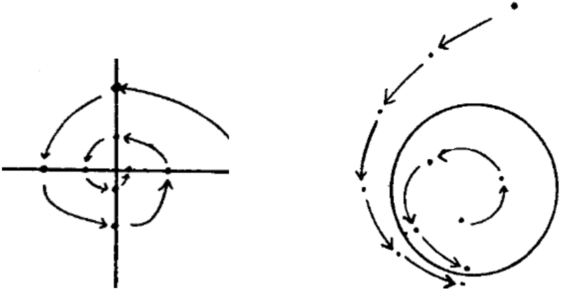
Example

$F_\lambda(r,\theta) = (r_1, \theta_1)$, where $r_1 = \lambda r + \beta r^3$ $(\beta < 0)$, $\theta_1 = \theta + \alpha$.



The origin is a fixed point, which is attracting for $0 < \lambda < 1$. For $\lambda > 1$, the origin is repelling and there is also an invariant circle $r = \sqrt{(1-\lambda)/\beta}$, which is an attractor.

Hopf bifurcation



The **Hopf bifurcation** occurs when a fixed point spawns an invariant (hyperbolic) cycle in transition between attracting and repelling behaviour. The Hopf bifurcation is **supercritical** if an attracting fixed point gives rise to an attracting cycle and **subcritical** if a repelling fixed point gives rise to a repelling cycle.

More examples

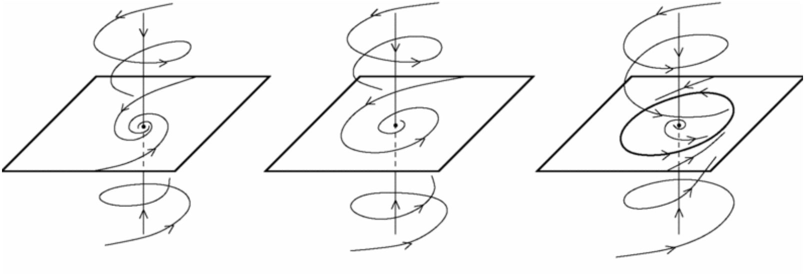
- $F_\lambda(r,\theta) = (r_1, \theta_1)$, where $r_1 = \lambda r + \beta r^3$, $\theta_1 = \theta + \alpha + \gamma r^2$.

  The restriction of F_λ to the invariant circle $r = \sqrt{(1-\lambda)/\beta}$ is a rotation. The angle of rotation depends on λ.

- $F_\lambda(r,\theta) = (r_1, \theta_1)$, where $r_1 = \lambda r + \beta r^3$, $\theta_1 = \theta + \alpha + \varepsilon \sin(k\theta)$.

  The restriction of F_λ to the invariant circle $r = \sqrt{(1-\lambda)/\beta}$ is a standard map.

Hopf bifurcation in dimension 3



Normal form

Let $F : \mathbb{R}^2 \to \mathbb{R}^2$ be a smooth map fixing the origin and consider it as a transformation of $\mathbb{C}$.

Theorem Suppose $F(z) = \mu z + O(|z|^2)$ as $z \to 0$, where $|\mu| = 1$ and $\mu^k \neq 1$ for any integer k, $1 \leq k \leq 5$. Then there exist neighborhoods U, W of 0 and a (real) diffeomorphism $L : U \to W$ such that $L^{-1} \circ F \circ L = \mu z + \beta z^2 \bar{z} + O(|z|^5)$ as $z \to 0$.

The map $L^{-1} \circ F \circ L$ is called the **normal form** of the map F at 0. The number β is called the **first Lyapunov coefficient** of F at 0.

More generally, if $\mu^k \neq 1$ for any integer k, $1 \leq k \leq 2\ell + 3$, then the diffeomorphism L can be chosen so that $L^{-1} \circ F \circ L = \mu z + \beta_1 |z|^2 z + \beta_2 |z|^4 z + \cdots + \beta_\ell |z|^{2\ell} z + O(|z|^{2\ell+3})$ as $z \to 0$. The numbers $\beta_1, \beta_2, \ldots$ are called **Lyapunov coefficients** of F at 0.

Hopf bifurcation: sufficient condition

Theorem Suppose $F_\lambda : \mathbb{R}^2 \to \mathbb{R}^2$ is a smooth family of maps satisfying the following conditions:

- $F_\lambda(\mathbf{0}) = \mathbf{0}$;
- DF_λ has complex conjugate eigenvalues $\mu(\lambda), \overline{\mu(\lambda)}$;
- $|\mu(0)| = 1$ and $(\mu(0))^k \neq 1$ for any integer k, $1 \leq k \leq 5$;
- $\frac{d}{d\lambda}|\mu(\lambda)| > 0$ at $\lambda = 0$;
- the first Lyapunov coefficient of F_0 at $\mathbf{0}$ satisfies $\beta < 0$.

Then there exist a neighborhood U of the origin, $\varepsilon > 0$, and a smooth closed curve ζ_λ defined for $0 < \lambda < \varepsilon$ such that **(i)** $F_\lambda(\zeta_\lambda) = \zeta_\lambda$, **(ii)** the curve ζ_λ is attracting for the map F_λ in U; **(iii)** in polar coordinates (r, θ), the curve ζ_λ is given by an equation $r = r_\lambda(\theta)$; **(iv)** $r_\lambda \to 0$ and $r'_\lambda \to 0$ uniformly as $\lambda \to 0$.

Remark. For the subcritical Hopf bifurcation, the last two conditions should be $\frac{d}{d\lambda}|\mu(\lambda)| < 0$ at $\lambda = 0$ and $\beta > 0$.