MATH 614

Dynamical Systems and Chaos

Lecture 29: Local holomorphic dynamics at fixed points.

Classification of periodic points

Let $U \subset \mathbb{C}$ be a domain and $F : U \to \mathbb{C}$ be a holomorphic function. Suppose that $F(z_0) = z_0$ for some $z_0 \in U$. The fixed point z_0 is called

- attracting if $|F'(z_0)| < 1$;
- repelling if $|F'(z_0)| > 1$;
- neutral if $|F'(z_0)| = 1$.

Now suppose that $F^n(z_1) = z_1$ for some $z_1 \in U$ and an integer $n \geq 1$. The periodic point z_1 is called

- attracting if $|(F^n)'(z_1)| < 1$;
- repelling if $|(F^n)'(z_1)| > 1$;
- neutral if $|(F^n)'(z_1)| = 1$.

The multiplier $(F^n)'(z_1)$ is the same for all points in the orbit of z_1. In particular, all these points are of the same type as z_1. Note that the multiplier is preserved under any holomorphic change of coordinates.

Hyperbolic fixed points

Theorem 1 Suppose z_0 is an attracting fixed point for a holomorphic function F. Then there exist $\delta > 0$ and $0 < \mu < 1$ such that

$$|F(z) - z_0| \leq \mu |z - z_0|$$

for any $z \in D = \{z \in \mathbb{C} : |z - z_0| < \delta\}$. In particular, $\lim_{n\to\infty} F^n(z) = z_0$ for all $z \in D$.

Theorem 2 Suppose z_0 is a repelling fixed point for a holomorphic function F. Then there exist $\delta > 0$ and $M > 1$ such that

$$|F(z) - z_0| \geq M |z - z_0|$$

for all $z \in D = \{z \in \mathbb{C} : |z - z_0| < \delta\}$. In particular, for any $z \in D \setminus \{z_0\}$ there is an integer $n \geq 1$ such that $F^n(z) \notin D$.

Theorem 3 Let F be a holomorphic function at 0 such that $F(0) = 0$ and $F'(0) = \lambda$, where $0 < |\lambda| < 1$. Then there is a neighborhood U of 0 and a holomorphic map $h : U \to \mathbb{C}$ such that $F \circ h = h \circ L$ in U, where $L(z) = \lambda z$.

Idea of the proof: We are looking for a map h of the form $h(z) = z + \sum_{i=2}^{\infty} c_i z^i$, where c_i are unknown coefficients. Let $F(z) = \lambda z + \sum_{i=2}^{\infty} a_i z^i$ be the Taylor expansion of F. The condition $F \circ h = h \circ L$ holds if

$$\lambda h(z) + \sum_{i=2}^{\infty} a_i (h(z))^i = \lambda z + \sum_{i=2}^{\infty} c_i \lambda^i z^i$$

or, equivalently,

$$\sum_{i=2}^{\infty} (\lambda^i - \lambda) c_i z^i = \sum_{i=2}^{\infty} a_i (h(z))^i.$$

From this equality of formal power series we can recursively determine all coefficients c_i. For example, $c_2 = a_2/(\lambda^2 - \lambda)$. Then one has to prove that the radius of convergence for the power series $h(z)$ is positive.

Theorem 4 Let F be a holomorphic function at 0 such that $F(0) = 0$ and $F'(0) = \lambda$, where $|\lambda| > 1$. Then there is a neighborhood U of 0 and a holomorphic map $h : U \to \mathbb{C}$ such that $F \circ h = h \circ L$ in $L^{-1}(U)$, where $L(z) = \lambda z$.

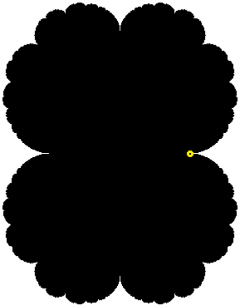
Idea of the proof: Since $F'(0) \neq 0$, the function F is invertible in a neighborhood of 0. The inverse function F^{-1} is also holomorphic. The point 0 is an attracting fixed point of F^{-1}.

It remains to apply the previous theorem.

Neutral fixed points

Example. • $F(z) = z + z^2$.

The map has a fixed point at 0, which is neutral: $F'(0) = 1$. The set D_0 of all points z satisfying $F^n(z) \to 0$ as $n \to \infty$ is open and connected.

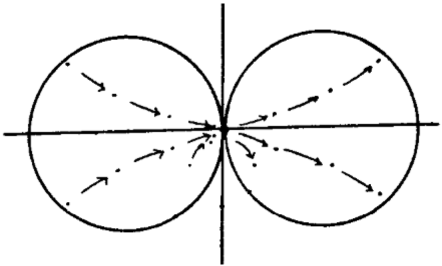


The fixed point 0 is one of the cusp points at the boundary of D_0. The others correspond to eventually fixed points.

Neutral fixed points

Proposition Suppose a function F is holomorphic at 0 and satisfies $F(0) = 0$, $F'(0) = 1$, $F''(0) = 2$ so that $F(z) = z + z^2 + O(|z|^3)$ as $z \to 0$.

Then there exists $\mu > 0$ such that **(i)** all points in the disc $D_- = \{z \in \mathbb{C} : |z + \mu| < \mu\}$ are attracted to 0; and **(ii)** all points in the disc $D_+ = \{z \in \mathbb{C} : |z - \mu| < \mu\}$ are repelled from 0.



Proof: We change coordinates using the function $H(z) = 1/z$, which maps the discs D_- and D_+ onto halfplanes $\mathrm{Re}\, z < -1/(2\mu)$ and $\mathrm{Re}\, z > 1/(2\mu)$.

The function F is changed to $G(z) = 1/F(1/z)$. Since $F(z) = z + z^2 + O(|z|^3)$ as $z \to 0$, it follows that

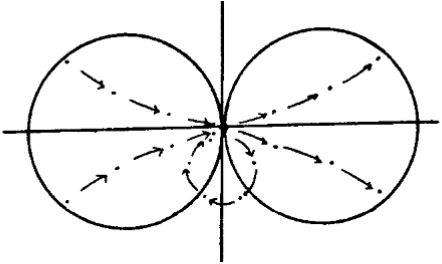
$$\begin{aligned} F(1/z) &= z^{-1} + z^{-2} + O(|z|^{-3}) \\ &= z^{-1}\big(1 + z^{-1} + O(|z|^{-2})\big) \quad \text{as} \quad z \to \infty. \end{aligned}$$

Then

$$\begin{aligned} G(z) &= z\big(1 + z^{-1} + O(|z|^{-2})\big)^{-1} \\ &= z\big(1 - z^{-1} + O(|z|^{-2})\big) = z - 1 + O(|z|^{-1}). \end{aligned}$$

If μ is small enough, then the halfplane $\mathrm{Re}\, z < -1/(2\mu)$ is invariant under the map G while the halfplane $\mathrm{Re}\, z > 1/(2\mu)$ is invariant under G^{-1}.

The proposition suggests that for most of the points in a neighborhood of 0, the forward and backward orbits under the map F both converge to 0.



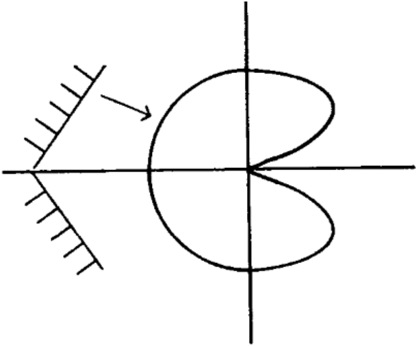
Examples. • $F(z) = \dfrac{z}{1-z}$.

This is a Möbius transformation with 0 the only fixed point. It follows that all forward and backward orbits converge to 0.

• $F(z) = z + z^2$.

The orbits of all points on the ray $z > 0$ converge to ∞ and so are the orbits of all points in a small cusp about this ray.

In the proof of the proposition, we could use wedge-shaped regions instead of halfplanes. This would allow to extend basins of attraction from discs to cardioid-shaped regions.



In the case not all points near 0 are attracted to 0, the set of points that are attracted is locally a simply connected domain with 0 on its boundary. This domain is called the **attracting petal** of the fixed point 0. Similarly, there is also the **repelling petal** of 0.